Another sufficient condition for twisting sliding mode controller construction

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***Abstract*—Sliding mode controller has wide applications in control engineering due to its robustness in the input channel, and the second order twisting sliding mode controller can efficiently solve the chattering problem invited by discontinuities in classical sliding mode controller. In this paper the control algorithm for twisting controller design is derived with a simplified strategy for parameters' choice. It is shown that an ideal or real second order sliding mode is established. A practical example, are presented using twisting controller design strategy, demonstrate the effectiveness of the proposed method.**

***Index Terms*—Sliding mode control, Lyapunov function**

I. Introduction

A. Sliding mode control

In practical world, the uncertainties and disturbances are generally existed in control system, and become a common issue for designing a robust controller. Hence, the sliding mode control technique (SMC) is considered the most viable approach: It reduce the order of dynamical system, constrain the state variable's motion on "sliding surface". Once the sliding mode is built, the rest of work is just to ensure that the internal dynamical system is stable.

Levant (1993) [1] give a basic concept for sliding mode control [1]: every motion that takes place strictly on the constraint manifold $\sigma = 0$ is called an ideal sliding. Every motion in a small neighbourhood of the manifold is called a real sliding.

Based on above definitions, for ideal sliding, SMC divides the whole control process into two stages. Suppose the form of dynamical system is like:

$$\dot{\mathbf{x}} = f(x, t, u) \tag{1}$$

For first-order sliding [1]: on the first stage (transient process), the control input u drives the state variable x to the sliding surface $\mathcal{S}$ in finite time:

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n | \sigma(\mathbf{x}, t) = 0\} \tag{2}$$

Where σ is called "sliding variable" (or constraint manifold, sliding output). Then on the second stage (steady state process), the control input u restrict state variable's motion on the sliding surface and keep it move along the sliding surface towards the origin.

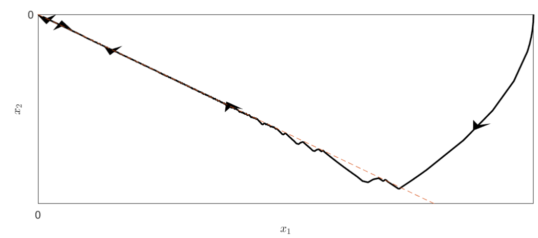


Fig. 1. Example phase portrait with sliding control input

Edwards (1998) [2] has shown the basic concept for designing a classical (first-order) sliding mode controller for single-input single-output system:

$$u = -\rho \ \mathrm{sign}(\sigma) \tag{3}$$

With proper choice of ρ and σ the robustness can been verified and this controller can completely reject the matched disturbances and minimized the unmatched disturbances, however in this case, the discontinuous control input u causes chattering problem in practice.

Levant(2003) [3] has basically present the concept for the second-order sliding mode controller:

For second-order sliding [3]: the whole control process keeps almost the same with above, where the sliding surface shows the differences:

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n | \sigma(\mathbf{x}, t) = \dot{\sigma}(\mathbf{x}, t) = 0\} \tag{4}$$

In this case, the control input u is designed as a differential function, where its derivative $\dot{u}$ is discontinuous, and only appears in the second derivative of σ. Hence, the chattering effect can be expected to be significantly attenuated.

Shtessel(2014) [4] basically shows how to design second-order twisting controller for the dynamical system $x = a(x,t) + b(x,t)v(t)$ with $\ddot{\sigma} = f(x,t) + g(x,t)v(t)$, where $|f(x,t)| \leq C$ and $0 < K_m \leq g(x,t) \leq K_M$:

$$\begin{aligned} \dot{u} &= v \\ v &= -r_1 \ \mathrm{sign}(\sigma) - r_2 \ \mathrm{sign}(\dot{\sigma}) \end{aligned} \tag{5}$$

with a sufficient condition for finite-time convergence of $\sigma, \dot{\sigma}$ (on transient process):

$$(r_1 + r_2)K_m - C > (r_1 - r_2)K_M + C \quad (6)$$
$$(r_1 - r_2)K_m - C > 0 \quad (7)$$

However, how to allocate the parameters r_1 and r_2 becomes a trouble in practical implementation when using the inequalities above to design twisting controller.

Based on their works above, this essay will finish two targets:

$$\begin{cases} \bullet \text{ Explore a simplified condition for sliding mode twisting controller design} \\ \bullet \text{ Stability analysis by Lyapunov candidate test.} \\ \bullet \text{ Discuss the best proportion for allocating } r_1 \text{ and } r_2 \end{cases} \quad (8)$$

B. Lyapunov candidate test

Shankar Sastry (1999) [5] has basically give methods for stability analysis using Lyapunov function. Moreover, Mobayen(2023) [6] mentioned that if $\dot{v} + mv^\alpha \le 0$ with $m > 0, \alpha \in (0,1)$ where v is positive definite function and decrescent function, then v will converge to zero in time t_s:

$$t_s = \frac{v(t_0)^{1-\alpha}}{m(1-\alpha)} \quad (9)$$

II. Problem Formulation

A. Problem explanation

For the dynamical system, suppose the form is like:

$$\dot{\mathbf{x}} = a(\mathbf{x}, t) + b(\mathbf{x}, t)u(t) \quad (10)$$

where $\mathbf{x}(t) \in R^n$, $u(t) \in R$ are state variable of system and control input respectively. Denote that u is designed to be differentiable equation which $\dot{u} = v$.

Then, we have that:

$$\dot{\sigma} = \sigma_{\mathbf{x}}(a + bu) + \sigma_t \quad (11)$$

and

$$\ddot{\sigma} = \dot{\sigma}_t + \dot{\sigma}_{\mathbf{x}}(a + bu) + \dot{\sigma}_u v \quad (12)$$
$$= \dot{\sigma}_t + \dot{\sigma}_{\mathbf{x}}(a + bu) + \sigma_{\mathbf{x}} bv \quad (13)$$

Then, by separation, let

$$f(\mathbf{x}, t) = \dot{\sigma}_t + \dot{\sigma}_{\mathbf{x}}(a + bu) \quad (14)$$
$$g(\mathbf{x}, t) = \sigma_{\mathbf{x}} b \quad (15)$$

By substituting (15) and (16) in (14), $\ddot{\sigma}$ can be rewritten as:

$$\ddot{\sigma} = f(\mathbf{x}, t) + g(\mathbf{x}, t)v(t) \quad (16)$$

By letting $\mathbf{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \sigma \\ \dot{\sigma} \end{bmatrix}$ The dynamical system for sliding variable σ is also considered as:

$$\begin{aligned} \dot{s_1}(\mathbf{x}, t) &= s_2(\mathbf{x}, t) \\ \dot{s_2}(\mathbf{x}, t) &= f(\mathbf{x}, t) + g(\mathbf{x}, t)v(t) \end{aligned} \quad (17)$$

Where $s_1, s_2 \in R$, if the following assumption is satisfied:

$$f \in [-C, C] \quad \text{(for some } C > 0) \quad \text{(A1)}$$
$$g \in [K_m, K_M] \quad \text{(for some } K_M > K_m > 0) \quad \text{(A2)}$$

Then, design sliding mode twisting controller as:

$$\begin{aligned} \dot{u} &= v \\ v &= -a(\text{sign}(s_1) + \frac{1}{p}\text{sign}(s_2)) \qquad (a > 0, p > 1) \end{aligned} \quad (18)$$

For above control law, a proper Lyapunov function is designed below to explore a simplified condition for control gain a and proportion p to achieve a finite-time convergence for $\mathbf{s}$. In addition, the "best proportion p" will be discussed to make sure that with the best proportion p, the amplitude of v can close to its infimum.

B. Stability analysis

Design the lyapunov function according to Shtessel(2017) [7] as:

$$v_0(s_1, s_2) = k_0^2 a^2 s_1^2 + \gamma |s_1|^{\frac{3}{2}} sign(s_1) s_2 + k_0 a |s_1| s_2^2 + \frac{1}{4} s_2^2 \quad (19)$$

with some $k_0 > 0$ and $\gamma > 0$ which will be defined later.
The whole stability analysis process is divided into three steps:
• v_0 is a positive definite function.
• v_0 is a decrescent function.
• $\dot{v_0}$ satisfies the relationship $\dot{v_0} + mv_0^\alpha \le 0$ for some $m > 0, \alpha \in (0,1)$.
Firstly, in order to say that "v_0 is a positive definite function.", we rewrite v_0 as:

$$v_0 = \mathbf{z}^T \mathbf{A} \mathbf{z} + \frac{1}{4} s_2^4 \quad (20)$$

where

$$\mathbf{z} = \begin{bmatrix} s_1 & |s_1|^{\frac{1}{2}} s_2 \end{bmatrix}^T \qquad \mathbf{A} = \begin{bmatrix} k_0^2 a^2 & \frac{\gamma}{2} \\ \frac{\gamma}{2} & k_0 a \end{bmatrix} \quad (21)$$

In order to get a positive definite matrix $\mathbf{A}$, a sufficient condition is to let both the trace of $\mathbf{A}$ and determinant of $\mathbf{A}$ be positive:

$$tr(\mathbf{A}) = k_0^2 a^2 + k_0 a \quad (22)$$
$$det(\mathbf{A}) = k_0^3 a^3 - \frac{\gamma^2}{4} \quad (23)$$

with above equation, the trace is always positive with positive k_0 and a. Besides, the determinant will be positive when setting $\gamma \in (0, 2(k_0 a)^{1.5})$. Thus, with $\gamma \in (0, 2(k_0 a)^{1.5})$, $\mathbf{A}$ is positive definite. Furthermore, in view of Rayleigh principle:

$$\lambda_{min}(\mathbf{A})||\mathbf{z}||_2^2 \le \mathbf{z}^T \mathbf{A} \mathbf{z} \le \lambda_{max}(\mathbf{A})||\mathbf{z}||_2^2 \quad (24)$$

Thus, eqs.(20) can be modified as:

$$v_0 \ge \lambda_{min}(\mathbf{A})(|s_1|^2 + |s_1||s_2|^2) + \frac{1}{4} s_2^4 \quad (25)$$
$$\ge \lambda_{min}(\mathbf{A})|s_1|^2 + \frac{1}{4} s_2^4 \quad (26)$$

Since $\mathbf{A} > 0 \Rightarrow \lambda_{min}(\mathbf{A}) > 0$, v_0 is a positive definite function.

Secondly, in order to say that v_0 is a decrescent function, we modify eqs(20) as:

$$
\begin{aligned}
v_0 &\le \lambda_{max}(\mathbf{A})(|s_1|^2 + |s_1||s_2|^2) + \frac{1}{4}s_2^4 && (27)\\
&\le \lambda_{max}(\mathbf{A})(|s_1|^2 + \frac{1}{2}|s_1|^2 + \frac{1}{2}s_2^4) + \frac{1}{4}s_2^4 && (28)\\
&= \frac{3}{2}\lambda_{max}(\mathbf{A})s_1^2 + \frac{1}{2}(\lambda_{max}(\mathbf{A}) + \frac{1}{2})s_2^4 && (29)\\
&= \mathbf{k}^T\mathbf{Q}\mathbf{k} && (30)
\end{aligned}
$$

where

$$
\mathbf{k} = \begin{bmatrix} s_1 & s_2^2 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} \frac{3}{2}\lambda_{max}(\mathbf{A}) & 0 \\ 0 & \frac{1}{2}(\lambda_{max}(\mathbf{A}) + \frac{1}{2}) \end{bmatrix} \tag{31}
$$

By positive definite matrix $\mathbf{A}$, $\mathbf{Q}$ is also positive definite. Then, by Rayleigh principle, (30) can be modified as:

$$
v_0 \le \lambda_{max}(\mathbf{Q})(s_1^2 + s_2^4) \tag{32}
$$

In view of inequality (Hardy, Littlewood, & G.polya, 1951) [8]:

$$
\begin{aligned}
&(\Sigma_{i=1}^n \alpha_i^s)^{\frac{1}{s}} \le (\Sigma_{j=1}^n \alpha_j^r)^{\frac{1}{r}} \\
&(\text{for any } 0 < r < s, \alpha_1, \alpha_2, ..., \alpha_n > 0)
\end{aligned} \tag{33}
$$

(32) can be modified as:

$$
v_0 \le \lambda_{max}(\mathbf{Q})(|s_1|^{\frac{1}{2}} + |s_2|)^4 \tag{34}
$$

Thus, v_0 is a decrescent function. Thirdly, compute the derivative $\dot{v}$:

$$
\begin{aligned}
\dot{v_0} = &(2k_0^2a^2s_1 + \frac{3}{2}\gamma|s_1|^{\frac{1}{2}}s_2 + k_0a \text{ sign}(s_1)s_2^2)\dot{s_1} + \\
&(\gamma|x|^{\frac{3}{2}} \text{ sign}(s_1) + 2k_0a|s_1|s_2 + s_2^3)\dot{s_2}
\end{aligned} \tag{35}
$$

substitute (18) into (35):

$$
\begin{aligned}
&\dot{v_0} = -\gamma|s_1|^{\frac{3}{2}}[ag + \frac{ag}{p} \text{ sign}(s_1s_2) - f \text{ sign}(s_1)] - 2k_0a|s_1| \\
&|s_2|[ag \text{ sign}(s_1s_2) + \frac{ag}{p} - f \text{ sign}(s_2) - k_0a \text{ sign}(s_1s_2)] - \\
&|s_2|^3[ag \text{ sign}(s_1s_2) + \frac{ag}{p} - f \text{ sign}(s_2) - k_0a \text{ sign}(s_1s_2)] + \\
&\frac{3}{2}\gamma|s_1|^{\frac{1}{2}}s_2^2
\end{aligned} \tag{36}
$$

Where

$$
\begin{aligned}
ag + \frac{ag}{p} \text{ sign}(s_1s_2) - f \text{ sign}(s_1) &\ge \frac{p-1}{p}ag - C \\
&\ge \frac{p-1}{p}aK_m - C
\end{aligned} \tag{37}
$$

and

$$
\begin{aligned}
&ag \text{ sign}(s_1s_2) + \frac{ag}{p} - f \text{ sign}(s_2) - k_0a \text{ sign}(s_1s_2) \\
&\ge -a|g - k_0| + \frac{ag}{p} - C \\
&= \min\{-\frac{p-1}{p}ag + ak_0 - C, \frac{p+1}{p}ag - ak_0 - C\} \\
&\ge \min\{-\frac{p-1}{p}aK_M + ak_0 - C, \frac{p+1}{p}aK_m - ak_0 - C\}
\end{aligned} \tag{38}
$$

By setting $d_1 = \frac{p-1}{p}aK_m - C$, $d_2 = \min\{-\frac{p-1}{p}aK_M + ak_0 - C, \frac{p+1}{p}aK_m - ak_0 - C\}$, (36) can be rewritten as:

$$
\dot{v_0} \le -\gamma|s_1|^{\frac{3}{2}}d_1 - \mathbf{w}^T\mathbf{P}\mathbf{w} \tag{39}
$$

where

$$
\mathbf{w} = \begin{bmatrix} |s_1|^{\frac{1}{2}}|s_2|^{\frac{1}{2}} & |s_2|^{\frac{3}{2}} \end{bmatrix} \tag{40}
$$

$$
\mathbf{P} = \begin{bmatrix} 2k_0ad_2 & -\frac{3}{4}\gamma \\ -\frac{3}{4}\gamma & d_2 \end{bmatrix} \tag{41}
$$

In order to let $\dot{v_0}$ be negative, a sufficient condition is to let d_1 be positive and $\mathbf{P}$ be positive definite, which is equivalent to let d_1, $tr(\mathbf{P})$, and $det(\mathbf{P})$ be positive:

$$
tr(\mathbf{P}) = (2k_0a + 1)d_2 \tag{42}
$$

$$
det(\mathbf{P}) = 2k_0ad_2^2 - \frac{9}{16}\gamma^2 \tag{43}
$$

From (39), in order to let d_1 be positive, we set:

$$
a > \frac{p}{p-1}\frac{C}{K_m} \tag{44}
$$

From (42) and (43), In order to let $tr(\mathbf{P}) > 0$ and $det(\mathbf{P}) > 0$, we need to let $d_2 > 0$ and $\gamma < \frac{4}{3}\sqrt{2k_0a}d_2$. On the one hand, combined with the previous condition $\gamma \in (0, 2(k_0a)^{1.5})$, we get the condition for choosing γ:

$$
\gamma \in (0, \min\{2(k_0a)^{1.5}, \frac{4}{3}\sqrt{2k_0a}d_2\}) \tag{45}
$$

On the other hand, in order to let d_2 be positive, we set:

$$
k_0 > \frac{p-1}{p}K_M + \frac{C}{a} \tag{46}
$$

$$
k_0 < \frac{p+1}{p}K_m - \frac{C}{a} \tag{47}
$$

In order to let k_0 be well-defined, we have that:

$$
\frac{p-1}{p}K_M + \frac{C}{a} < \frac{p+1}{p}K_m - \frac{C}{a} \tag{48}
$$

$$
\Rightarrow a[\frac{p+1}{2p}K_m - \frac{p-1}{2p}K_M] > C \tag{49}
$$

From (46), (47), we set k_0 as the midpoint of $(\frac{p-1}{p}K_M + \frac{C}{a}, \frac{p+1}{p}K_m - \frac{C}{a})$:

$$
k_0 = \frac{p+1}{2p}K_m + \frac{p-1}{2p}K_M \tag{50}
$$

Thus, if conditions (44), (45), (49), (50) is satisfied, $\mathbf{P}$ would be positive definite and $d_2 > 0$. Then

$$\dot{v_0} \leq -\gamma|s_1|^{\frac{3}{2}} d_2 - \lambda_{min}(\mathbf{P})(|s_1||s_2| + |s_2|^3) \quad (51)$$
$$\leq -\gamma d_2|s_1|^{\frac{3}{2}} - \lambda_{min}(\mathbf{P})|s_2|^3 \quad (52)$$
$$\leq -K(|s_1|^{\frac{3}{2}} + |s_2|^3) \quad (53)$$

where

$$K = min\{\lambda_{min}(\mathbf{P}), \gamma d_2\} \quad (54)$$

in view of inequality (Hardy, Littlewood, & G.polya, 1951) [8]:

$$(\frac{1}{n}\Sigma_{i=1}^{n}\alpha_i^r)^{\frac{1}{r}} \leq (\frac{1}{n}\Sigma_{j=1}^{n}\alpha_j^s)^{\frac{1}{s}} \quad \text{(for any } 0 < r < s, \alpha_1, \alpha_2, ..., \alpha_n > 0) \quad (55)$$

(53) can be modified as:

$$\dot{v_0} \leq -\frac{K}{4}[|s_1|^{\frac{1}{2}} + |s_2|]^3 \quad (56)$$

Combine (34) with (56):

$$\dot{v_0} \leq -\frac{K}{4(\lambda_{max}(Q))^{\frac{3}{4}}} v_0^{\frac{3}{4}} \quad (57)$$

Thus, with eqs.(57), it's sufficient to claim that finite-time convergence is achieved by such designed sliding mode twisting controller, and an ideal sliding mode or real sliding mode is established.1

C. best proportion p

Next step is to discuss the relationship between the amplitude of v which is denoted by r and the proportion:

$$r = a(1 + \frac{1}{p}) \quad (58)$$

and find the value of p so that r can be close to its minimum. Firstly, remark the two inequalities from above:

$$a[\frac{p+1}{2p}K_m - \frac{p-1}{2p}K_M] > C \quad (59)$$
$$a > \frac{p}{p-1}\frac{C}{K_m} \quad (60)$$

Note that $\frac{p+1}{2p}K_m - \frac{p-1}{2p}K_M$ must be positive by requirement, so we have an extra condition for p:

$$(p+1)K_m > (p-1)K_M \Rightarrow p < \frac{K_m + K_M}{K_M - K_m} \quad (61)$$

we conclude that

$$a > \max_{p\in(1,\frac{K_m+K_M}{K_M-K_m})} \left\{ \frac{C}{[\frac{p+1}{2p}K_m - \frac{p-1}{2p}K_M]}, \frac{C}{\frac{p-1}{p}K_m} \right\} \quad (62)$$

which follows that

$$r > \max_{p\in(1,\frac{K_m+K_M}{K_M-K_m})} \left\{ \frac{C}{\frac{1}{2}K_m - \frac{p-1}{2(p+1)}K_M}, \frac{C}{\frac{p-1}{p+1}K_m} \right\} \quad (63)$$

Now, our aim is to find the minimum of right side by proper p. If we denote that:

$$c = \frac{p-1}{p+1} \qquad (0 < c < \frac{K_m}{K_M}) \quad (64)$$
$$h(c) = \frac{C}{[\frac{1}{2}K_m - \frac{c}{2}K_M]} \quad (65)$$
$$q(c) = \frac{C}{cK_m} \quad (66)$$

Then (63) can be rewritten as:

$$r > \max_{c\in(0,\frac{K_m}{K_M})} \{h(c), q(c)\} \quad (67)$$

it's easy to see that when c is increasing, $h(c)$ increases and q(c) decreases, in addition, $\lim_{c\to 0} q(c) \to \infty$. Thus, if $h(c) = q(c)$, the right side of (67) reaches its minimum:

$$\begin{aligned} &f(c) = q(c) \\ \Rightarrow &\frac{1}{2}K_m - \frac{c}{2}K_M = cK_m \\ \Rightarrow &c = \frac{K_m}{K_M + 2K_m} \\ \Rightarrow &p = \frac{K_M + 3K_m}{K_m + K_M} \end{aligned} \quad (68)$$

Combined with (62), (63), we have that

$$r > \frac{(K_M + 2K_m)}{K_m^2} C \quad (69)$$
$$a > \frac{3K_m + K_M}{2K_m^2} C \quad (70)$$

Remark 1. *Since "the best proportion p" is determined by K_m, K_M, moreover, K_m, K_M are determined by $g = \sigma_{\mathbf{x}} b$. Therefore, we say that the "best proportion p" is determined by $\sigma_{\mathbf{x}}$, where σ is a designed sliding variable.*

III. MAIN RESULT

Theorem 1. *For dynamical system (10) for $\mathbf{x}$, and dynamical system (17) for $\mathbf{s}$, if the assumption (A1) and (A2) hold, the sliding mode twisting controller can be designed as :*

$$\begin{aligned} \dot{u} &= v \\ v &= a(\ sign(s_1) + \frac{K_m + K_M}{3K_m + K_M}\ sign(s_2)) \end{aligned} \quad (71)$$

with condition:

$$a > \frac{3K_m + K_M}{2K_m^2} C \quad (72)$$

With the control law (71), and condition (72), finite-time convergence of $\mathbf{s}$ can be achieved, and the convergence time t_s can be estimated as:

$$t_s \leq \frac{16(\lambda_{max}(\mathbf{Q}))^{\frac{3}{4}} v_0(t_0)^{\frac{1}{4}}}{K} \quad (73)$$

where

$$v_0 = k_0^2 a^2 s_1^2 + \gamma|s_1|^{\frac{3}{2}} sign(s_1)s_2 + k_0 a|s_1|s_2^2 + \frac{1}{4}s_2^2 \quad (74)$$

with

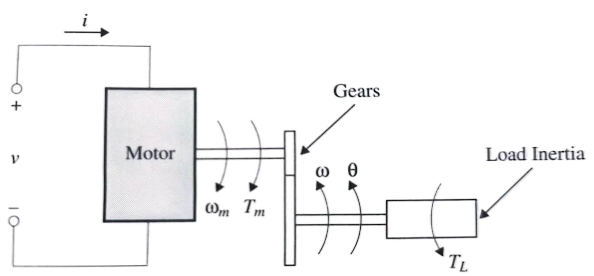
$$\begin{aligned}&\gamma \in (0, \min\{2(k_0 a)^{\frac{3}{2}}, \frac{4}{3}\sqrt{2k_0 a}d_2\}\\ &K = min\{\lambda_{min}(\mathbf{P}), \gamma d_2\}\end{aligned} \tag{75}$$

where

$$\begin{aligned}&k_0 = \frac{2K_m + K_M}{3K_m + K_M}K_m + \frac{K_m}{3K_m + K_M}K_M\\ &d_2 = \min\{-\frac{2K_m}{3K_m + K_M}aK_M + ak_0 - C,\\ &\frac{4K_m + 2K_M}{3K_m + K_M}aK_m - ak_0 - C\}\\ &\mathbf{Q} = \begin{bmatrix}\frac{3}{2}\lambda_{max}(\mathbf{A}) & 0\\ 0 & \frac{1}{2}(\lambda_{max}(\mathbf{A}) + \frac{1}{2})\end{bmatrix}\\ &\mathbf{P} = \begin{bmatrix}2k_0 a d_2 & -\frac{3}{4}\gamma\\ -\frac{3}{4}\gamma & d_2\end{bmatrix}\end{aligned} \tag{76}$$

Moreover, when the control gain "a" tends to $\frac{3K_m+K_M}{2K_m^2}C$, the amplitude of v (denoted by "r") tends to its infimum: $\frac{K_M+2K_m}{K_m^2}C$

IV. Practical example



Considering the problem of controlling the angular position of the shaft in a DC motor. Assume $T_L = 0$, The process of construction of dynamic equations for θ, ω, i are similar as Belanger [9], and are omitted:

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\\ i\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & \frac{Nk_m}{J_e}\\ 0 & -\frac{Nk_m}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}\theta\\ \omega\\ i\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{L}\end{bmatrix}v \tag{77}$$

with relationship

$$J_e = J + N^2 J_m \tag{78}$$

Where armature voltage v is the control input, The gear ratio N is the ratio of angles and velocities of the two shafts; The torques have the same ratio. R and L are the resistance and inductance of the armature circuit respectively. The motor drives a load with moment of inertia J, and the rotor of the dc motor has inertia J_m.

To design a continuous control input v, claim that

$$v = v_0 + \int_0^t w(\tau)d\tau \tag{79}$$

A. Analysis

For the specific values according to Belanger [9]: $k_m = 0.05\,Nm/A$, $R = 1.2\,\Omega$, $L = 0.05\,H$, $J_m = 0.0008\,kg\,m^2$, $J = 0.020kg\,m^2$ and $N = 12$, Design the sliding surface which is the same as Edwards [2]: set

$$x = \begin{bmatrix}\theta\\ \omega\\ i\end{bmatrix} \qquad S = \begin{bmatrix}0.9013 & 0.8563 & 1\end{bmatrix} \tag{80}$$

$$\mathcal{S} = \{(\theta, \omega, i) \in \mathbb{R}^3 | \sigma_1 = S * x = 0\} \tag{81}$$

For the sliding variable $\sigma_1 = S * x$, denote that $A = \begin{bmatrix}\theta\\ \omega\\ i\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & \frac{Nk_m}{J_e}\\ 0 & -\frac{Nk_m}{L} & -\frac{R}{L}\end{bmatrix}$ is the system matrix, and $B = \begin{bmatrix}0\\ 0\\ \frac{1}{L}\end{bmatrix}$ is the input distribution matrix. Then the system equations can be rewritten as :

$$\dot{x} = Ax + Bv \tag{82}$$

For the sliding variable $s_1 = S*x$, the corresponding dynamic system for $\mathbf{s} = [s_1, s_2]^T$ becomes

$$\dot{s_1}(t) = s_2(t) = S(Ax(t) + Bv(t)) \tag{83}$$

$$\dot{s_2}(t) = S[A(Ax(t) + Bv(t)) + Bw(t)] \tag{84}$$

Then, to apply twisting controller design strategy, estimate the parameter C, K_m, K_M as following:

$$C = ||SAA||||x(t)||_{max} + ||SAB||||v(t)||_{max} \tag{85}$$

$$K_m = K_M = ||SB|| \tag{86}$$

With the system above, set the proportion p:

$$p = \frac{3K_m + K_M}{K_m + K_M} = 2 \tag{87}$$

and

$$w = -a(sign(\sigma_1) + \frac{1}{2}sign(\sigma_2)) \tag{88}$$

Set the initial value $x = [1, 0, 0]^T$, then we have initial value for $\mathbf{s}$:

$$s_1(0) = 0.9013 > 0 \tag{89}$$

$$s_2(0) = 20v(0) \tag{90}$$

In order to limit the amplitude of v, choose initial value of $v(0) = v_0 = -0.1$ such that $s_2(0) < 0$, then initial value $w(0) = w_0$ becomes

$$w_0 = -a * (1 - \frac{1}{2}) = -\frac{1}{2}a \tag{91}$$

This will make the control input v changes in a "slow mode" for a period of time at the beginning until $s_1 s_2 > 0$. Assume that during the control process, the $||x(t)||_{max} \leq 2||x(0)|| = 2$, $||v(t)||_{max} \leq 2||v(0)|| = 2$ then according to theorem 1, the twisting controller is designed as:

$$w = -(\frac{4||SAA|| + 4||SAB||}{||SB||})(sign(s_1) + \frac{1}{2}sign(s_2)) \tag{92}$$

B. Simulation result

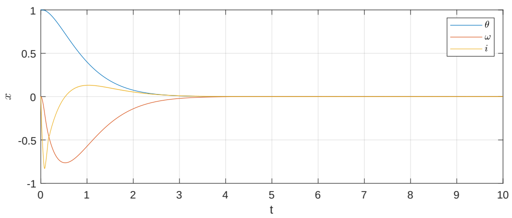


Fig. 2. states θ, ω, i

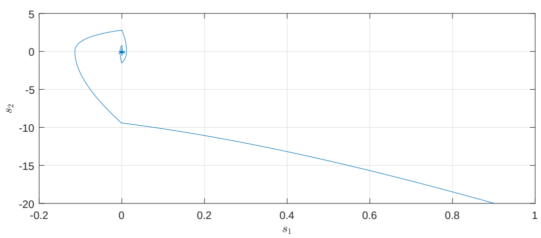
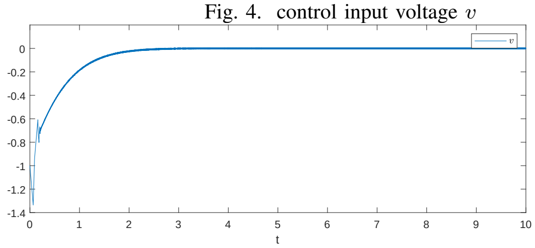


Fig. 3. phase portrait of s_1 and s_2

From the above picture, the state variables and sliding variables converge to zero finally.

Fig. 4. control input voltage v



Compare to the results of Edwards [2] in which the classical sliding mode control strategy is applied, the amplitude of control input voltage v here has been effectively reduced from -2 to 0.

V. Conclusion

This essay basically discuss the details for construction of ”twisting” controller, mainly focus on the simplifying the sufficient condition given by Shtessel(2014) [4], and find the best proportion p. During the process, the amplitude of u can be reduced.

However, this is the single-input case, for more general case like multi-input multi-sliding output case, the corresponding proper Lyapunov function for stability analysis should be reconsidered. Furthermore, in practical use, generally the estimation of the parameter C are difficult, this is also a predicament for reducing the amplitude of control input.

Next step this method is considered to be extended to the multi-input case where more details of designing ”twisting” controller will be explored. The research was partly supported by XJTLU RDF-22-01-050.

References

[1] A. Levant, “Sliding order and sliding accuracy in sliding mode control,” *International journal of control*, vol. 58, no. 6, pp. 1247–1263, 1993.

[2] C. Edwards and S. Spurgeon, *Sliding mode control: theory and applications*. Crc Press, 1998.

[3] A. Levant, “Higher-order sliding modes, differentiation and output-feedback control,” *International journal of Control*, vol. 76, no. 9-10, pp. 924–941, 2003.

[4] Y. Shtessel, C. Edwards, L. Fridman, A. Levant *et al.*, *Sliding mode control and observation*. Springer, 2014, vol. 10.

[5] S. Sastry and S. Sastry, “Lyapunov stability theory,” *Nonlinear Systems: Analysis, Stability, and Control*, pp. 182–234, 1999.

[6] S. Mobayen, F. Bayat, S. ud Din, and M. T. Vu, “Barrier function-based adaptive nonsingular terminal sliding mode control technique for a class of disturbed nonlinear systems,” *ISA transactions*, vol. 134, pp. 481–496, 2023.

[7] Y. B. Shtessel, J. A. Moreno, and L. M. Fridman, “Twisting sliding mode control with adaptation: Lyapunov design, methodology and application,” *Automatica*, vol. 75, pp. 229–235, 2017.

[8] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*. Cambridge university press, 1952.

[9] P. R. Belanger, *Control Engineering: A Modern Approach*, 1st ed. USA: Oxford University Press, Inc., 1995.