Homework 1 — Math 5323
Due: Fridays, 28 Jan 2022, by 11:59 p.m. CDT

Question 4.1: Show that $\mathcal{F} = \{(a, \infty) : a \in \overline{\mathbb{R}}\}$, with the convention that $(\infty, \infty) = \varnothing$ is a topology on $\mathbb{R}$ which is T_0 but not T_1. (We denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$)

Proof. By Definition 4.1.1, to show that the collection $\mathcal{F}$ is a topology on $\mathbb{R}$, we are required to show that

(i) $\varnothing \in \mathcal{F}$, $\mathbb{R} \in \mathcal{F}$. This is because by the convention $(\infty, \infty) = \varnothing \in \mathcal{F}$ as $\infty \in \overline{\mathbb{R}}$, and $\mathbb{R} = (-\infty, \infty) \in \mathcal{F}$ as $-\infty \in \overline{\mathbb{R}}$.

(ii) $\mathcal{F}$ is closed under the (arbitrary) union. We choose a collection $\{U_\alpha \in \mathcal{F} : \alpha \in I\}$ of $\mathcal{F}$. By Definition of $\mathcal{F}$, we can write

$$U_\alpha = (a_\alpha, \infty)$$

for some $a_\alpha \in \overline{\mathbb{R}}$. We have that by Definition of intervals and Definition of $\mathcal{F}$

$$\bigcup_{\alpha \in I} U_\alpha = (b, \infty) \in \mathcal{F}$$

where we denote $b := inf\{a_\alpha : \alpha \in I\} \in \overline{\mathbb{R}}$

(iii) $\mathcal{F}$ is closed under the finite intersection. We choose a finite elements $U_1, \cdots, U_n \in \mathcal{F}$ for some $n \in \mathbb{N}$. By Definition of $\mathcal{F}$, we can write

$$U_i = (a_i, \infty)$$

for some $\alpha_1, \cdots, \alpha_n \in \overline{\mathbb{R}}$. We have that by Definition of intervals and Definition of $\mathcal{F}$

$$\bigcap_{i=1}^{n} U_i = (b, \infty) \in \mathcal{F}$$

where we denote $b := \max\{\alpha_i : i = 1, \cdots, n\} \in \mathbb{R} \subset \overline{\mathbb{R}}$.

We choose $x, y \in \mathbb{R}$ with $x \neq y$. Without loss of generality, we can assume $x < y$. By the completeness of the real numbers, we have $z := \frac{x+y}{2} \in \mathbb{R} \subset \overline{\mathbb{R}}$. Hence we have $x \notin (z, \infty)$, $y \in (z, \infty)$ and $(z, \infty) \in \mathcal{F}$. By Definition 4.1.4, we proved that the topology $\mathcal{F}$ is T_0. To disprove that $\mathcal{F}$ is T_1, it is enough to give a counter-example. We consider $1, 2$. We cannot find an element (a, ∞) of $\overline{\mathbb{R}}$ such that

$$1 \in (a, \infty) \text{ but } 2 \notin (a, \infty).$$

This is true since we have the implication

$$1 \in (a, \infty) \text{ implies that } 2 \in (a, \infty).$$

by the properties of the real numbers. By Definition 4.1.4, we proved that $\mathcal{F}$ is not T_1. $\square$

Question 4.2: Let $\mathcal{F}$ be the collection of subsets of $\mathbb{R}^2$ containing $\varnothing$, $\mathbb{R}^2$, and all the complements of finite number of lines and points. Prove that $\mathcal{F}$ is a topology on $\mathbb{R}^2$ which is T_2 but not Hausdorff.

Proof. By the condition, we have for any $A \subset \mathbb{R}^2$

$$A \in \mathcal{F} \text{ if and only if either } A = \varnothing,\ A = \mathbb{R}^2,\ A^c = \bigcup_{i=1}^{n} \{a_i : a_i \in \mathbb{R}^2 \text{or } a_i = (x_i, y_i) \text{ with } y_i = m_i x_i + n_i\}$$

for some $n \in \mathbb{N}$ and some $m_i, n_i \in \mathbb{R}$.

(i) $\varnothing, \mathbb{R}^2 \in \mathcal{F}$ is true without doubt.

(ii) We choose $\{A_i\}_{i\in I} \subset \mathcal{F}$. If there is one A_i is $\mathbb{R}^2$, there is nothing to prove. We consider the case $A_i \neq \mathbb{R}^2$ for all $i \in I$. If $A_i \neq \varnothing$ for all $i \in I$, there is nothing to prove since the arbitrary union of empty sets is still empty. Now we have the statement: there exists $p \in I$ such that

$$\text{either } A_p^c = \bigcup_{i=1}^{n} \{a_i : a_i \in \mathbb{R}^2\} \text{ or } A_p^c = \bigcup_{i=1}^{n} \{(x_i, y_i) \in \mathbb{R}^2 : y_i = m_i x_i + n_i\}$$

for some $n \in \mathbb{N}$ and some $m_i, n_i \in \mathbb{R}$. Now by De Morgan's law, we have

$$(\bigcup_{i\in I} A_i)^c = \bigcap_{i\in I} A_i^c \subset A_p^c.$$

Since we have the subset of finite number of line and points is still a finite number of line and points, we have $(\bigcup_{i\in I} A_i)^c$ is a finite number of lines and points, which immediately implies that $\bigcup_{i\in I} A_i \in \mathcal{F}$.

(iii) We prove $\mathcal{F}$ is closed under a finite n intersection by induction on n. There is nothing to prove for the base step $n = 1$. For the inductive step, by the induction hypothesis, we have $B := \bigcap_{i=1}^{n} A_i \in \mathcal{F}$ and by the condition, $A := A_{n+1} \in \mathcal{F}$. We denote $C := B \cap A$. Then $C = \bigcap_{i=1}^{n+1} A_i$ and by De Morgan's law $C^c = B^c \cup A^c$. If each of B and A is the empty set or the full space, this case is trivial. Now we consider both B^c and A^c are a finite number of lines and points. Since the finite union of finite number of lines and points is still the finite number of lines and points, we immediately have that C^c is a finite number of lines and points hence by Definition of $\mathcal{F}$, $\bigcap_{i=1}^{n+1} A_i = C \in \mathcal{F}$.

□

Question 4.3: Let X be a metric space with the metric ρ and $F \subset X$ be closed. For any $x \in X$, define

$$\rho(x, F) = \inf\{\rho(x, y) : y \in F\}$$

(a) Prove that $\rho(x, F) = 0$ if and only if $x \in F$.

(b) Prove that $U = \{x \in X : \rho(x, F) < c\}$ is an open set containing F for any $c > 0$.

(c) Prove that X is normal.

Proof. (a) We want to prove that $\rho(x, F) = 0$ implies that $x \in F$. By Definition of inf, we can choose a sequence $y_i \in F$ such that $\lim_{i\to\infty} \rho(x, y_i) = 0$ which implies that

$$y_i \text{ approach } x \text{ in the metric space } X.$$

By Definition of closeness, we immediately have that $x \in X$. We want to prove that $x \in F$ implies that $\rho(x, F) = 0$. This is so obvious by definition of metric spaces, we have $\rho(x, x) = 0$ and $\rho(z, y) \geq 0$ for any $z, y \in X$.

(b) We choose $c > 0$ randomly. By (a), we have $F \subset U$ since for any $x \in F$, $\rho(x, F) = 0$ and $0 < c$. Now we prove that U is open. By Definition of a topology, it is equivalent to prove that $X \setminus U$ is closed. We choose a sequence $x_k \in X \setminus U$ such that $\rho(x_k, x) \to 0$ for some $x \in X$. By Definition of closeness, it required to prove that $x \in X \setminus U$. Then by Definition of U and Lemma 0.1, for each $k \in \mathbb{N}$ such that

$$c \leq \rho(x_k, F) \leq \rho(x, x_k) + \rho(x, F),$$

after pushing both sides to ∞, due to $\lim_{k\to\infty} \rho(x, x_k) = 0$, we have

$$c \leq \rho(x, F).$$

and by Definition of U, we have $x \in X \setminus U$.

(c) By Definition (e)4.1.4, we firstly need to prove that it is T_1. We choose $x, y \in X$ with $x \neq y$ randomly. By Definition of metric spaces, we have $d := \rho(x, y) > 0$. Now we consider $W := \{z \in X : \rho(x, z) < \frac{d}{2}\}$. W is obviously open since it is an open ball as $W = B(x, \frac{d}{2})$. Also it is obviously $x \in W$ as $\rho(x, x) = 0$ by Definition of metric spaces. It is also true $y \notin W$ by our Definition of d. Similarly we can also find open set V such that $y \in V$ and $x \notin V$. Since such x, y was chosen randomly , by (b) of Definition 4.1.4, we proved that X is T_1. We choose randomly two closed sets A and B such that $A \cap B = \varnothing$. We consider $U_A := \big\{x \in X : \rho(x, A) < \rho(x, B)\big\}$ and $U_B := \big\{x \in X : \rho(x, A) > \rho(x, B)\big\}$. There is no doubt that $U_A \cap U_B = \varnothing$ and there is no doubt that $A \subset U_A$ and $B \subset U_B$ and trust that mathematicians use this Definition to catch the geometrical meaning well. Now we want to prove that U_A and U_B are both open. We prove that U_A is open and similarly we can prove that U_B is open. Now by Definition of a topology, it is equivalent to prove that $X \setminus U_A$ is closed. We choose a sequence $y_k \in X \setminus U_A$ such that $\rho(y_k, y) \to 0$ for some $y \in X$. By Definition of closeness, we are required to prove that $y \in X \setminus U_A$. Now we have

$$\rho(y, B) \leq \rho(y, y_k) + \rho(y_k, B) \leq \rho(y, y_k) + \rho(y_k, A) \leq \rho(y, A) + \rho(y_k, y) + \rho(y_k, y)$$

where the first and the last is due to Lemma 0.1, the second is due Definition of U_A, after pushing both sides into ∞ and using $\rho(y_k, y) \to 0$ as $n \to \infty$, we have $\rho(y, B) \leq \rho(y, A)$ and by Definition of U_A, we have $y \in X \setminus U_A$.

□

Lemma 0.1. *Let X be a metric space with $x, y \in X$ and $B \subset X$ be a closed set. Prove*

$$\rho(x, B) \leq \rho(x, y) + \rho(y, B).$$

Proof. By Definition of inf, we can choose $z_k \in B$ such that $\rho(y, B) = \lim_{k\to\infty} \rho(y, z_k)$. Then by Definition of inf with $z_k \in B$ and triangle inequalities, we have for each

$$\rho(x, B) \leq \rho(x, z_k) \leq \rho(x, y) + \rho(y, z_k),$$

after pushing both sides into ∞, we have

$$\rho(x, B) \leq \rho(x, y) + \lim_{k\to\infty} \rho(y, z_k) = \rho(x, y) + \rho(y, B)$$

□

Question 4.4: Prove that in a Hausdorff space X, every singleton set $\{x\}$ is closed.

Proof. We are not interested in the empty space and the space contains the only point or two points. We choose $x \in X$ randomly. By Definition of closeness, it is equivalent to prove that $X \setminus \{x\}$ is open. For each $a \in X \setminus \{x\}$, by Definition of the set operation, we know that $a \neq x$ and hence by Definition of Hausdorff space, we have a pair of disjoint open sets U_a and V_a such that

$$a \in U_a \text{ and } x \in V_a.$$

We denote $W := \bigcup_{a \in X \setminus \{x\}} U_a$. Then W is open by Definition of topological space and each U_a is open and it is obviously that $X \setminus \{x\} \subset W$. Now for each $a \in X \setminus \{x\}$, $x \in V_a$ and $U_a \cap V_a = \varnothing$ implies that $x \notin U_a$, which says that $U_a \subset X \setminus \{x\}$ and hence we have $W \subset X \setminus \{x\}$. So far we proved that $X \setminus \{x\} = W$ is open.

□

Question 4.5: Let

$$\mathcal{F} = \{U \cup (V \cap \mathbb{Q}) : U, V \text{ open in Euclidean topology } \mathbb{R}\}$$

Prove that $\mathcal{F}$ is a topology on $\mathbb{R}$ which is Hausdorff but not regular.

Proof. (i) By Definition of $\mathcal{F}$ and Definition of topology spaces, we have

$$\varnothing = \varnothing \cup (\varnothing \cap \mathbb{Q}) \in \mathcal{F} \text{ and } \mathbb{R} = \mathbb{R} \cup (\varnothing \cap \mathbb{Q}) \in \mathcal{F}.$$

(ii) We choose a collection $\{A_i\}_{i \in I} \subset \mathcal{F}$ randomly. By Definition of $\mathcal{F}$, for each $i \in I$, we can write in the following form

$$A_i = U_i \cup (V_i \cap \mathbb{Q})$$

for some open sets U_i, V_i. Then by De Morgan's law and Definition of topology spaces

$$\bigcup_{i \in I} A_i = \bigcup_{i \in I} U_i \cup (V_i \cap \mathbb{Q}) = (\bigcup_{i \in I} U_i) \cup ((\bigcup_{i \in I} V_i) \cap \mathbb{Q}) \in \mathcal{F}$$

(iii) We choose a finite number $A_1, \cdots, A_n \in \mathcal{F}$ randomly. By Definition of $\mathcal{F}$, for each $1 \leq i \leq n$, we can write in the following form

$$A_i = U_i \cup (V_i \cap \mathbb{Q})$$

for some open sets U_i, V_i. Then by De Morgan's law and Definition of topology spaces

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} U_i \cup (V_i \cap \mathbb{Q}) = (\bigcap_{i=1}^{n} U_i) \cup ((\bigcap_{i=1}^{n} V_i) \cap \mathbb{Q}) \in \mathcal{F}.$$

Now we prove $(\mathbb{R}, \mathcal{F})$ is Hausdorff. We choose $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$ randomly. We consider

$$B(x_1, r) = B(x_1, r) \cup (B(x_1, r) \cap \mathbb{Q}) \in \mathcal{F} \text{ and } B(x_2, r) = B(x_2, r) \cup (B(x_2, r) \cap \mathbb{Q}) \in \mathcal{F}$$

due to Definition of $\mathcal{F}$ and $B(x_1, r), B(x_2, r)$ open in $\mathbb{R}$ where we denote $r := \frac{1}{4}|x_1 - x_2| > 0$ by Definition of norms. There is no doubt that $x_1 \in B(x_1, r)$ and $x_2 \in B(x_2, r)$. If $z \in B(x_1, r) \cap B(x_2, r)$, then by the triangle inequality, we have

$$|x_1 - x_2| \leq |x_1 - z| + |x_2 - z| < 2r = \frac{1}{2}|x_1 - x_2|$$

which gives us a perfect contradiction. So we proved that $B(x_1, r) \cap B(x_2, r) = \varnothing$ and due to (c)Definition4.1.4, we proved that $\mathcal{F}$ is Hausdorff. We disprove $\mathcal{F}$ is regular by counterexample. We consider $A := \mathbb{Q}^c$. Now $\mathbb{Q} = \varnothing \cap (\mathbb{R} \cap \mathbb{Q}) \in \mathcal{F}$ and we proved that $\mathcal{F}$ is a topology. So A is closed. We consider $1 \in \mathbb{Q}$. We choose $B_1, B_2 \in \mathcal{F}$ randomly with $B_1 \cap B_2 = \varnothing$. Then by Definition of $\mathcal{F}$, we can write $B_1 = U_1 \cup (V_1 \cap \mathbb{Q})$ and $B_2 = U_2 \cup (V_2 \cap \mathbb{Q})$ for some open sets U_1, U_2, V_1 and V_2. If $1 \notin B_1$, then we are done. So we consider the case $1 \in B_1$. (we need to argue it in more explicit way)

□

Homework 2 — Math 5323
Due: Fridays, 4 Feb 2022, by 11:59 p.m. CDT

Question 4.8: Let X be a topological space, $A \subset X$ be closed, and $g \in C(A)$ with $g = 0$ on ∂A. Prove that

$$G(x) = g(x) \text{ for } x \in A \text{ and } G(x) = 0 \text{ otherwise,}$$

is continuous on X.

Proof. We choose $K \subset \mathbb{R}$ closed randomly. By Definition 4.1.5 and $G^{-1}(X \setminus K) = X \setminus G^{-1}(K)$, it is equivalent to prove that $G^{-1}(K)$ closed in X. We prove this in 2 cases. We consider the first case, i.e. $0 \in K$. Then by Definition of G, we have $G^{-1}(K) = g^{-1}(K) \cup (int(A))^c$. This is true since for any $x \in G^{-1}(K)$, if $x \in A$, then $x \in g^{-1}(K)$, if $x \notin A$, then either $x \in A^c$ or $x \in \partial A$ and $(intA)^c = (A^c \cup \partial A)$ and since for any $x \in A^c \cup \partial A \cup g^{-1}(K)$, if $x \in A^c$, then $G(x) = 0$ and hence $x \in G^{-1}(K)$ since $0 \in K$, if $x \in \partial A$, then $x \in G^{-1}(K)$ since $0 \in K$ and definition of g and if $x \in g^{-1}(K)$, then $g(x) \in K$ and $x \in A$ implies that $x \in G^{-1}(K)$. Now since K is closed and $g \in C(A)$, by Definition 4.1.5, we have $g^{-1}(K)$ is closed in A. By Definition of relative topologies, we know $g^{-1}(K) = A \cap E$ for some E closed in X. Then by Definition of topologies, we know $G^{-1}(K)$ is closed. We consider the second case, i.e. $0 \notin K$. Since $0 \notin K$ implies that $G^{-1}(K) \subset A$, $G^{-1}(K) = g^{-1}(K)$ and we proved that $g^{-1}(K)$ is closed in X. So we are done. $\square$

Question 4.9: Let X be a topological space, Y be a Hausdorff space, and $f, g \in C(X, Y)$. Prove that

(a) $\{x \in X : f(x) = g(x)\}$ is closed, and

(b) If $f = g$ on a dense subset of X, then $f = g$ on X.

Proof. (a) We denote $A := \{x \in X : f(x) = g(x)\}$. We choose a sequence $x_k \in A$ with $x_k \to x$ for some $x \in X$. Now by Definition of closeness, it is equivalent to prove that $x \in A$. By Definition of A, it is equivalent to prove that $f(x) = g(x)$. Since $x_k \in A$, by Definition of A, we have $f(x_k) = g(x_k)$ which implies that $f(x_k) - g(x_k) = 0$. Since $C(X, Y)$ is an algebra implying that $f - g \in C(X, Y)$, we have

$$f(x) - g(x) = (f - g)(x) = \lim_{k\to\infty} (f(x_k) - g(x_k)) = 0$$

which implies that $f(x) = g(x)$ immediately.

(b) We denote A be such a dense subset of X such that $f = g$ on A. We choose $x \in X$ randomly. By Definition of dense subsets, we can choose a sequence $x_k \in A$ such that $\lim_{k\to\infty} x_k = x$. Now by the condition $f, g \in C(X, Y)$ we have

$$f(x) - g(x) = \lim_{k\to\infty} f(x_k) - \lim_{k\to\infty} g(x_k) = \lim_{k\to\infty} (f(x_k) - g(x_k)) = 0$$

where the last equality is due to $f = g$ on A, which implies that $f(x) = g(x)$. But such $x \in X$ was chosen randomly. So we proved that $f = g$ on X.

□

Comment: Where do we need the Hausdorff condition posed on the target space?

Question 4.10: Let $(\mathbb{R}, \mathcal{F}_c)$ and $(\mathbb{R}, \mathcal{F}_e)$ be the cofinite and Euclidean topological spaces on $\mathbb{R}$, respectively. Prove that every continuous function $f : (\mathbb{R}, \mathcal{F}_c) \to (\mathbb{R}, \mathcal{F}_e)$ is a constant function.

Proof. We choose a continuous function $f : (\mathbb{R}, \mathcal{F}_c) \to (\mathbb{R}, \mathcal{F}_e)$. We argue it by contradiction and suppose that f is not constant, then we assume f has at least 2 values p and q with $p \neq q$. Since $\mathbb{R}$ is Hausdorff, we can choose two disjoint open sets U and V such that $p \in U$, $q \in V$ and $U \cap V = \varnothing$. Since f is continuity, by Definition 4.1.5, we have $f^{-1}(U)$ and $f^{-1}(V)$ open and $f^{-1}(U) \cap f^{-1}(V) = \varnothing$ by Definition of preimages. Then we have

$$\mathbb{R} \setminus f^{-1}(U) \cup \mathbb{R} \setminus f^{-1}(V) = \mathbb{R} \setminus (f^{-1}(U) \cap f^{-1}(V)) = \mathbb{R}. \tag{1}$$

Now by Definition of confinite topologies and the union of finite sets is still finite, $\mathbb{R} \setminus f^{-1}(U) \cup \mathbb{R} \setminus f^{-1}(V)$ is finite and this contradicts with that $\mathbb{R}$ is infinite due to (1).

□

Question 4.11: Let X be a Hausdorff space. Prove that the following statements are equivalent.

(i) X is normal

(ii) For any disjoint closed sets A, B, there exists a $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.

(iii) Any $f \in C(A, [a,b])$ with A closed can be extended to a function $F \in C(X, [a,b])$ such that $F\big|_A = f$.

Proof. We finish this proof by the following process

$$\begin{tikzcd} & \text{(i)} \arrow[dl] & \\ \text{(ii)} \arrow[rr] & & \text{(iii)} \arrow[ul] \end{tikzcd}$$

We prove $(i) \Rightarrow (ii)$ by Urysohn's Lemma. We choose closed sets $A, B \subset X$ with $A \cap B = \varnothing$ randomly. Since X is normal, by Theorem 4.1.10, we have $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.

We prove $(iii) \Rightarrow (i)$. We choose A, B closed in X with $A \cap B = \varnothing$. We define $f_1 : A \to [0,1]$ on $f_1(x) = 0$ and define $f_2 : B \to [0,1]$ on $f_2(x) = 1$. Since any constant function is continuous, by the given condition, we have $F_1 \in C(X, [0,1])$ and $F_2 \in C(X, [0,1])$ such that $F_1\big|_A = f_1$ and $F_2\big|_B = f_2$. We define $F := F_1 + F_2$ and there is no doubt that F is continuous since the sum of continuous functions is still continuous. Now $V := F^{-1}([0, \frac{1}{4}))$ is open in X and $A \subset F^{-1}([0, \frac{1}{4}))$ since F is continuous. Similarly $U := F^{-1}((\frac{3}{4}, 1])$ is open in X and $B \subset F^{-1}((\frac{3}{4}, 1])$. There is no doubt that $V \cap U = \varnothing$. Then by Definition 4.1.4(e), we proved that X is normal since such A, B is chosen randomly and the Hausdorff condition is stronger than T_1.

We prove $(ii) \Rightarrow (iii)$. (details needed to be done)

$\square$

Question 4.12: Prove that a topological space X is Hausdorff if and only if the limit of any convergent net is unique.

Proof. First we prove $\Rightarrow$ direction. We choose a net $A \to X; \alpha \mapsto x_\alpha$ with $x_\alpha \to x$ as $\alpha \to \infty$ for some $x \in X$. We argue this by contradiction and suppose $x, y \in X$ with $x \neq y$ such that $x_\alpha \to x$ and $y_\alpha \to y$. Since X is Hausdorff, by Definition 4.1.4, we have disjoint open sets U, V such that $x \in U$ and $y \in V$. Since $x_\alpha \to x$, by Definition of $\to$, there exists $\beta \in A$ such that for any $\alpha \in A$

$$\beta \preceq \alpha \Rightarrow x_\alpha \in U. \tag{2}$$

Similarly we can find $\gamma \in A$ such that for any $\alpha \in A$

$$\gamma \preceq \alpha \Rightarrow x_\alpha \in V. \tag{3}$$

Then by Definition 4.2.1 and Definition 4.2.2, we have $\delta \in A$ such that $\beta \preceq \delta$ and $\gamma \preceq \delta$. Then by (1) and (2), we have

$$x_\delta \in U \cap V$$

which immediately implies that $U \cap V \neq \varnothing$. This contradicts with our choice of U and V.

Second we prove $\Leftarrow$ direction. We argue this by contradiction and by Definition 4.1.4, we can choose points $x, y \in X$ with $x \neq y$ such that for all open sets U, V with $x \in U$ and $y \in V$, $U \cap V \neq \varnothing$. We define the directed set

$$A = \{(U, V) : U, V \text{ are open and } x \in U, y \in V\}$$

with the partial order

$$(U_1, V_1) \preceq (U_2, V_2) \Leftrightarrow U_1 \supseteq U_2 \text{ and } V_1 \supseteq V_2$$

The checking that this is a well-defined directed set is left for readers. The only interesting part is (iii) in Definition 4.2.1. By axiom of choice, we can choose a net $(x_{(U,V)})_{(U,V) \in A}$ in X such that for each $(U, V) \in A$, $x_{(U,V)} \in U \cap V$. Now by the condition, it is enough to prove that $x_{(U,V)} \to x$ and $x_{(U,V)} \to y$. Now for any open set U_0 of X with $x \in U_0$, we have by Definition 4.2.1

$$(U_0, X) \preceq (U, V) \Rightarrow x_{(U,V)} \in U \cap V \subseteq U \subseteq U_0,$$

which implies that $x_{(U,V)} \to x$ by Definition of $\to$. Similarly we can prove that $x_{(U,V)} \to y$ and hence we achieve a contradiction. $\square$

Homework 3 — Math 5323

Due: Fridays, 11 Feb 2022, by 11:59 p.m. CDT

Question 4.14: Prove that every sequentially compact space is countably compact.

Proof. We denote X be a sequentially compact space. We denote $\{A_i\}_{i\in\mathbb{N}}$ be a countable cover of X. We argue this by contradiction. Then by Definition 4.3.14, for each $n \in X$, $A_1 \cup \cdots \cup A_n \subsetneq X$. Then we have a sequence $x_n \in X$ such that

$$x_n \notin A_m \text{ for all } m \leq n. \tag{1}$$

Then by Definition 4.3.14, we have a convergent sequence x_{n_p} with a limit $x \in X$. By Definition of covers, we have some $l \in \mathbb{N}$ such that $x \in A_l$. Since $x_{n_p} \to x$ as $p \to \infty$, we can choose $l \in \mathbb{N}$ such that $x_{n_l} \in A_l$. By Definition of subsequences, we have $l \leq n_l$. So by (1), we have $x_{n_l} \notin A_l$. So far we got a contradiction and hence we finished the proof.

□

Question 4.15: Let X be a topological space and $E \subset X$. Prove that a set $K \subset E$ is relatively compact in E if and only if it is compact in X.

Proof. We want to prove that K is relatively compact in E implies that K is compact in X. We choose an open cover $\{U_i\}_{i\in I}$ of K. Then by Definition 4.3.1 we have

$$K \subset \bigcup_{i\in I} U_i$$

Then we have by $K \subset E$ and De morgan's law

$$K = K \cap E \subset E \cap \bigcup_{i\in I} U_i = \bigcup_{i\in I}(E \cap U_i)$$

Then by Definition 4.1.2, 4.3.1 and 4.3.2, there exists finite number $i_1, \cdots, i_n$ such that

$$K \subset \bigcup_{l=1}^{n}(E \cap U_{i_l}) \subset \bigcup_{l=1}^{n} U_{i_l}$$

Then by Definition 4.3.2, we proved that K is compact in X.
We want to prove that K is compact in X implies that K is relatively compact in E. We choose an open cover $\{A_i\}_{i\in I}$ of K in E. Then By Definition 4.3.1 we have

$$K \subset \bigcup_{i\in I} A_i$$

and by Definition 4.1.2, we have for each i

$$A_i = U_i \cap E$$

for some U_i open in X. Then we have by De morgan's law

$$K \subset \bigcup_{i\in I}(U_i \cap E) = E \cap \bigcup_{i\in I} U_i \subset \bigcup_{i\in I} U_i.$$

Then by Definition 4.3.2, we have finite number $i_1, \cdots, i_n$ such that

$$K \subset \bigcup_{l=1}^{n} U_{i_l}$$

Then we have by $K \subset E$ and De morgan's law

$$K = K \cap E \subset (\bigcup_{l=1}^{n} U_{i_l}) \cap E = \bigcup_{l=1}^{n}(U_{i_l} \cap E) = \bigcup_{l=1}^{n} A_{i_l}$$

Then by Definition 4.3.2, we proved that K is relatively compact in E since A_{i_l} open in E. ☐

Question 4.17: Let X be a locally compact Hausdorff space and $K \subset X$ be compact, and $U \supset K$ be a precompact open set.

(a) Prove that every $f \in C(K)$ can be extended to a function $g \in C(\overline{U})$ such that $g = 0$ on $\overline{U} \setminus U$.

(b) Prove that every $f \in C(K)$ can be extended to a function $F \in C(X)$ such that $F = 0$ on U^c.

Proof. (a) We choose $f \in C(K)$ randomly. Since X is a locally compact Hausdorff space and K compact, U open with $K \subset U$, by Theorem 4.4.4, we have $h \in C(X, [0,1]) \subset C(X)$ such that $h = 1$ on K and $h = 0$ on $X \setminus A$ for some compact set A with $A \subset U$. Since X is a locally compact Hausdorff space with K compact, there exists $F \in C(K)$ with $F\big|_K = f$ on K. There is no doubt $hF \in C(X)$ since $C(X)$ is an algebra. We denote $g := hF\big|_{\overline{U}}$. There is no doubt $g \in C(\overline{U})$ since the restriction of continuous maps is still continuous. Now it remains to prove that

$$g = f \text{ on } K \text{ and } g = 0 \text{ on } \overline{U} \setminus U$$

Now for $x \in K$, $g(x) = h(x)F(x) = 1 \cdot f(x)$ by our choice of h and F. Now it remains to prove $g = hF\big|_{\overline{U}} = 0$ on $\overline{U} \setminus U$. Since $h = 0$ on $X \setminus A$, it is enough to prove $\overline{U} \setminus U \subset X \setminus A$ by Definition of restriction of functions. But this is so obviously since $A \subset U$ implies $X \setminus U \subset X \setminus A$ and U open implies that $\overline{U} \setminus U = \partial U \subset X \setminus U$

(b) We choose $f \in C(K)$ randomly. By (a), we have $g \in C(\overline{U})$ such that $g = 0$ on $\partial \overline{U}$. Now we define

$$G(x) = g(x) \text{ for } x \in \overline{U} \text{ and } G(x) = 0 \text{ otherwise.}$$

Then by Question 4.8 with $\overline{U}$ closed, we have $G \in C(X)$. Now it is enough to prove that $G = 0$ on $X \setminus U$. Since U is open, by Definition 4.1.1, we have

$$X = U \sqcup \partial U \sqcup X \setminus \overline{U}.$$

Then $X \setminus U = \partial U \sqcup X \setminus \overline{U}$. For $x \in X \setminus U$, if $x \in \partial U$, $G(x) = g(x) = 0$ since $\partial U = \partial \overline{U}$ and if $x \in X \setminus \overline{U}$, $G(x) = 0$ by Definition of G. So we proved that G is the desired extension.

$\square$

Question 4.18: (One-point compactification.) Let $(X, \mathcal{F})$ be a non-compact topological space. Pick any element not in X and denote it by ∞. Define

$$X^* = X \cup \{\infty\},\ \mathcal{F}^* = \mathcal{F} \cup \{X \setminus K \cup \{\infty\} : K \subset X \text{ is compact in } (X, \mathcal{F}) \text{ and } X \setminus K \in \mathcal{F}\}$$

Prove that

(a) $\mathcal{F}^*$ is a topology on X^* such that $(X, \mathcal{F})$ is the relative topological space of $(X^*, \mathcal{F}^*)$ and

(b) $(X^*, \mathcal{F}^*)$ is a compact space.

Proof. (a) First we prove that $\mathcal{F}^*$ is a topology on X^*.

(i) $\varnothing \in \mathcal{F}^*$ by Definition 4.1.1 and Definition of $\mathcal{F}^*$. $X^* = X \setminus \varnothing \cup \{\infty\} \in \mathcal{F}^*$ by Definition of $\mathcal{F}^*$ and Definition 4.1.1 with $\varnothing$ compact in $(X, \mathcal{F})$ and $X \setminus \varnothing \in \mathcal{F}$.

(ii) We choose $A_i \in \mathcal{F}^*$ with $i \in I$ randomly. We denote $A := \bigcup_{i \in I} A_i$. We prove $A \in \mathcal{F}^*$ in two separate cases. In case one, we consider there exists $j \in I$ such that $\infty \in A_j$. Then by Definition of $\mathcal{F}^*$, $A_j = X \setminus K_j \cup \{\infty\}$ for some $K_j \subset X$ compact. Then we find

$$A = (A \setminus \{\infty\}) \sqcup \{\infty\} = X \setminus K \sqcup \{\infty\}$$

where we denote $X \setminus K := A \setminus \{\infty\}$ for some $K \subset X$. Now by Definition of $\mathcal{F}^*$, it remains to show that K is compact. But from Definition of A and with little set operations, we find that $K \subset K_j$. Now since the subset of a compact set is still compact, we are done in this case. In case two, we consider for any $i \in I$, $\infty \notin A_i$. Then by Definition of $\mathcal{F}^*$, we know that $A_i \in \mathcal{F}$ for any $i \in I$. Since $\mathcal{F}$ is a topology, by definition of topologies, there is no doubt $A \in \mathcal{F}$ and hence $A \in \mathcal{F}^*$.

(iii) We choose $A_1, \cdots, A_n \in \mathcal{F}^*$ randomly. We denote $A := \bigcap_{i=1}^{n} A_i$. We prove $A \in \mathcal{F}^*$ in separate cases. In case one, we consider that for all $1 \leq i \leq n$, $\infty \in A_i$. Then by Definition of $\mathcal{F}^*$, we have that for each $1 \leq i \leq n$, $A_i = X \setminus K_i \sqcup \{\infty\}$ for some $K_i \subset X$ compact. Then after little set operations, we have

$$A = X \setminus (\bigcup_{i=1}^{n} K_i) \sqcup \{\infty\}.$$

Now by Definition of $\mathcal{F}^*$, it remains to prove that $\bigcup_{i=1}^{n} K_i$ is compact. And this is true since the finite union of compacts sets is still compact. So we are done with the case one. Then by Definition of $\mathcal{F}^*$, we have $A_j \in \mathcal{F}$. In case two, we consider that there exists some $1 \leq j \leq n$ such that $\infty \notin A_j$. We denote $I := \{1 \leq i \leq n : \infty \notin A_i\}$ and $J := \{1 \leq i \leq n : \infty \in A_i\}$. Then for $i \in J$, we write $A_i = X \setminus K_i \sqcup \{\infty\}$ for some $K_i \subset X$ compact. Then we have

$$A = (\bigcap_{i \in I} A_i) \cap (\bigcap_{i \in J} X \setminus K_i).$$

sorry I dont know how to argue it furthermore $A \in \mathcal{F}$ without condition that $X \setminus K_i \in \mathcal{F}$

Now we want to prove that $(X, \mathcal{F})$ is the relatively topology space of $(X^*, \mathcal{F}^*)$. There is no doubt that $X \subset X^*$. We choose $U \in \mathcal{F}$ randomly. Now by Definition 4.1.2, it is enough to prove that $U = X \cap V$ for some $V \in \mathcal{F}^*$. But we have

$$U = U \cap X$$

since $U \subset X$ and $U \in \mathcal{F}^*$ by Definition of $\mathcal{F}^*$ with $U \in \mathcal{F}$.

(b) We choose $A_i \in \mathcal{F}^*$ be an open cover of X^*. Then by Definition 4.3.1, we have

$$X^* = \bigcup_{i \in I} A_i.$$

Since $\infty \in X^*$, we have $\infty \in A_p$ for some $p \in I$. Then by Definition of $\mathcal{F}^*$, we have

$$A_p = X \setminus K \cup \{\infty\}$$

for some K compact in $(X, \mathcal{F})$. We denote $J := \{i \in I : \infty \in A_i\}$. Then we have

$$K \subset X^* \subset \bigcup_{i \in J} A_i \setminus \{\infty\} \cup \bigcup_{i \in I \setminus J} A_i$$

By Definition of $\mathcal{F}^*$ and Definition 4.3.1, we know that $\{A_i \setminus \{\infty\} : i \in J\} \cup \{A_i : i \in I \setminus J\}$ is an open cover of K. Then by Definition 4.3.2, we have

$$K \subset \bigcup_{l=1}^{n} A_{i_l} \setminus \{\infty\} \cup \bigcup_{l=1}^{p} A_{i_l}.$$

for some $i_1, \cdots, i_n \in J$ and some $i_1, \cdots, i_p \in I \setminus J$. Then we have

$$X^* = X \setminus K \cup \{\infty\} \cup K = A_p \cup \bigcup_{i=1}^{n} A_{i_l} \setminus \{\infty\} \cup \bigcup_{l=1}^{p} A_{i_l}.$$

By Definition 4.3.1, we found an finite subcover of X^* and hence by Definition 4.3.2, we proved the result.

□

Question 4.19: Keeping notations in Question 4.18, prove that the space X^* is Hausdorff if and only if X is locally compact and Hausdorff.

Proof. First we want to prove that X^* is Hausdorff implies that X is locally compact and Hausdorff.

We prove that X is Hausdorff. We choose $x, y \in X$ with $x \neq y$. By Question 4.18, we have $x, y \in X^*$ with $x \neq y$. Since X^* is Hausdorff, by Definition 4.1.4, we have $U, V \in \mathcal{F}^*$ with $x \in U$ and $y \in V$ and $U \cap V = \varnothing$. by Question 4.19, we have $U, V \in \mathcal{F}$ where $x \neq \infty$ and $y \neq \infty$. Then By Definition 4.1.4,

We prove that X is locally compact. We choose $x \in X$ randomly. By Question 4.18, we know that $x \neq \infty$. Also we have $x, \infty \in X^*$. Since X^* is Hausdorff, by Definition 4.1.4, we have $U, V \in \mathcal{F}^*$ with $x \in U$, $\infty \in V$ and $U \cap V = \varnothing$. Now since $\infty \in V$, by Definition of $\mathcal{F}^*$, we have $V = X \setminus K \cup \{\infty\}$ for some K compact in X and since $\infty \in V$ with $U \cap V = \varnothing$, by Definition of $\mathcal{F}^*$, we have $U \in \mathcal{F}$. Now by Definition 4.4.1, it is enough to prove that

$$U \subset K.$$

which is equivalent to $X \setminus K \subset X \setminus U$ and this is proved quickly by $U \cap V = \varnothing$ implying $V \subset X \setminus U$ and $X \setminus K \subset V$.

Second we want to prove that X is locally compact and Hausdorff implies that X^* is Hausdorff. We choose $x, y \in X^*$ with $x \neq y$. If $x, y \in X$, then since X is Hausdorff, by Definition 4.1.4, we have $x \in U$, $y \in V$ and $U \cap V = \varnothing$ for some $U, V \in \mathcal{F}$. Then by Definition of $\mathcal{F}^*$, we have $U, V \in \mathcal{F}^*$. Then by Definition 4.1.4, we proved that X^* is Hausdorff. If one of y, x is ∞, with loss of generality, we assume $x \in X$ and $y = \infty$. Since X is locally compact, by Definition 4.4.1, we have K compact in X and $U \in \mathcal{F}$ such that $x \in U \subset K$. Since X is locally compact Hausdorff, with K compact and X open, we have a precompact set V such that

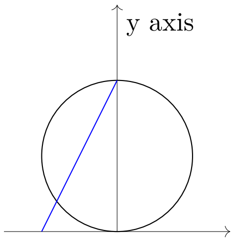
$$U \subset K \subset \overline{V},$$

where we denote $\overline{V}$ to be the closure of V in $(X, \mathcal{F})$, and hence $X \setminus \overline{V} \in \mathcal{F}$, which implies that $U \cap X \setminus \overline{V} = \varnothing$ and hence $U \cap (X \setminus \overline{V} \cup \{\infty\}) = \varnothing$. Since V is precompact, by Definition 4.3.2 we know $\overline{V}$ is compact and hence by Definition of $\mathcal{F}^*$, $U, X \setminus \overline{V} \cup \{\infty\} \in \mathcal{F}^*$. By Definition 4.1.4, we immediately proved that X^* is Hausdorff. ☐

Homework 4 — Math 5323
Due: Fridays, 18 Feb 2022, by 11:59 p.m. CDT

Question 4.21: Prove that in Euclidean topology, the one-point compactification of $\mathbb{R}$ is homeomorphic to the unit circle S^1 in $\mathbb{R}^2$

Proof. Using the idea of stereographic projection and moving up the unit circle, we consider the following picture



We denote $S^1 := \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = 1\}$. With the knowledge of plane geometry, we consider the map $f : S^1 \to \mathbb{R}^*$ defined by $(x,y) \mapsto \frac{2x}{2-y}$ for $(x,y) \neq (0,2)$ and $(x,y) \mapsto \infty$. There is no doubt that this is well-define since $(x,y) \neq (0,2)$ implies that $y \neq 2$. There is nothing needed to prove that f is injective since we can solve equations $x_1^2 + (y_1-1)^2 = 1$, $x_2^2 + (y_2-1)^2 = 1$ and $\frac{2x_1}{2-y_1} = \frac{2x_2}{2-y_2}$ by parameterization (x,y) by $(\cos\phi, 1+\sin\phi)$. Also the blue line indicates its subjectivity perfectly. So far we proved that f is bijective. There is nothing to doubt that S^1 is closed and bounded since the boundary is always is closed as $S^1 = \partial\{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 \leq 1\} = \partial B((0,1),1)$ and $S^1 \subset \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 < 2\} = B((0,1),2)$. Then by Heine–Borel theorem, we have S^1 is compact since we are only interested in Euclidean topology. Since $\mathbb{R}$ is locally compact Hausdorff as $x \in B(x,1) \subset \overline{B}(x,1)$, by Alexandroff Theorem, we know that $\mathbb{R}^*$ is Hausdorff. Now by Proposition 4.3.12, it is enough to prove that f is continuous, i.e. the preimage of open sets is still open. For U open in $\mathbb{R}^*$, by Definition 4.21, we have either $U \subset \mathbb{R}$ open in $\mathbb{R}$ or $U = \mathbb{R} \setminus K \cup \{\infty\}$ for some K compact in $\mathbb{R}$. In the first case $f^{-1}(U) = g^{-1}(U)$ where $g := f\big|_{S^1 \setminus \{(0,2)\}}$ and it is open since it is made by elementary functions and elementary functions are always continuous. In the second case by elementary set operations, we have $f^{-1}(U) = S^1 \setminus f^{-1}(K) \cup \{(0,2)\}$. Then we have by De morgan's law $S^1 \setminus f^{-1}(U) = f^{-1}(K) \cap S^1 \setminus \{(0,2)\} = f^{-1}(K)$. Now by Definition 4.1.1, it is enough to prove that

$$C := f^{-1}(K) \text{ closed in } S^1.$$

. We choose $x_k \in C$ with $x_k \to x$ for some $x \in S^1$. Now by property of closeness, it is enough to prove that

$$x \in C \text{ .i.e. } f(x) \in K$$

By our choice of x_k and the continuity of f, we have $f(x_k) \in K$ and $f(x_k) \to f(x)$ for some $f(x) \in \mathbb{R}^*$. Since K is compact in $\mathbb{R}$, by sequentially compactness, we have a convergent subsequence $f(x_{k_p})$ with $f(x_{k_p}) \to y$ for some $y \in \mathbb{R}$. By the uniqueness of the limit in the Hausdorff space $\mathbb{R}^*$, we must have $y = f(x) \in \mathbb{R}$. Now K is closed in $\mathbb{R}$. We must have $f(x) \in K$.

$\square$

Question 4.22: Let U be an open subset of a compact Hausdorff space X and U^* be the one-point compactification of U. Prove that the function $f : X \to U^*$ defined by

$$f(x) = x \text{ for } x \in U \text{ and } f(x) = \infty \text{ for } x \in X \setminus U$$

is continuous.

Proof. We denote $(U^*, \mathcal{F}^*)$ be the one-point compactification of U. By Definition 4.4.8 and Definition 4.1.2, we have

$$U^* = U \sqcup \{\infty\} \text{ and } \mathcal{F}^* = \mathcal{F} \sqcup \{U \setminus K \cup \{\infty\} : K \subset U \text{ is compact and } U \setminus K \in \mathcal{F}\}.$$

where we denote $\mathcal{F} := \{V \subset U : V \text{ open in } X\}$. We choose $V \in \mathcal{F}^*$ randomly. By Definition of continuous maps, it is enough to prove that $f^{-1}(V)$ is open in X. If $V \in \mathcal{F}^*$, then by Definition of $\mathcal{F}^*$, we have

$$\text{either } V \in \mathcal{F} \text{ or } V = U \setminus K \cup \{\infty\}$$

for some $K \subset U$ compact in U and $X \setminus K \in \mathcal{F}$. In the first case, by Definition of $\mathcal{F}$, we have $V \subset U$ and V open in X.Then by Definition of f, $f^{-1}(V) = V$ open in X. In the second case, we write

$$V = U \setminus K \cup \{\infty\}$$

By Definition 4.1,1, it is enough to prove that $X \setminus f^{-1}(V)$ is closed in X. Since K is compact in X and X is Hausdorff, by Proposition 4.3.7, we have K is closed in X. So now it is enough to prove that

$$X \setminus f^{-1}(V) = K$$

which is equivalent to show that

$$K \cap f^{-1}(V) = \varnothing \text{ and } f^{-1}(V) \cup K = X$$

Now by Definition of f and $K \subset U$, if $x \in K \cap f^{-1}(V)$, then $x \in K$ and $x = f(x) \in U \setminus K \cup \{\infty\}$ implying $f(x) \in U \setminus K$, which gives us a perfect contradiction. Now it remains to prove that $X \subset f^{-1}(V) \cup K$, which is equivalent to prove the statement for $x \in X$

$$x \in f^{-1}(V) \text{ or } x \in K$$

which is equivalent to prove the statement for $x \in X \setminus K$

$$f(x) \in V$$

by Definition of V, which is equivalent to prove the statement for $x \in X \setminus K$,

$$f(x) \in U \setminus K \text{ or } f(x) = \infty$$

Given $x \in X \setminus K$, if $x \in U$, then by Definition of f, $x = f(x) \in U \cap X \setminus K = U \setminus K$, if $x \in X \setminus U$, then by Definition of f, $f(x) = \infty$. ☐

Question 4.23: (The Stone-Weierstrass Theorem for noncompact spaces.) Let X be a noncompact locally compact Hausdorff space, and $\mathcal{A}$ be a subalgebra of $C_0(X,\mathbb{R})$ that separates points. Prove that either $\overline{\mathcal{A}} = C_0(X,\mathbb{R})$ or $\overline{\mathcal{A}} = \{f \in C_0(X,\mathbb{R}) : f(x_0) = 0\}$ for some $x_0 \in X$. (Hint: If there is an x_0 such that $f(x_0) = 0$ for all $f \in \mathcal{A}$), then consider the one-point compactification of $X \setminus \{x_0\}$ with $\infty = x_0$. Otherwise consider the one-point compactification of X.)

Proof. We prove the statement

$$\text{either } \overline{\mathcal{A}} = C_0(X,\mathbb{R}) \text{ or } \overline{\mathcal{A}} = \{f \in C_0(X,\mathbb{R}) : f(x_0) = 0\}.$$

We prove this statement by considering it in two separate cases. In the first case, we consider $f(x_0) = 0$ for all $f \in \mathcal{A}$ for some $x_0 \in X$. We consider the one-point compactification of $X \setminus \{x_0\}$ with $\infty := x_0$. By Definition 4.4.8, we have that Y^* is compact where $Y := X \setminus \{x_0\}$. Before we use the Stone-Weierstrass Theorem for compact Hausdorff spaces, i.e. Corollary 4.5.7, we need to check that $\mathcal{A}$ is a subalgebra of $C(X)$. There is no doubt that $\mathcal{A} \subset C(Y^*)$. Also there is no doubt that $\mathcal{A}$ is a subalgebra since $C_0(X,\mathbb{R}) \subset C(Y^*)$. As a set, X is identical to Y^*, there is doubt that $\mathcal{A}$ separates points Y^* by the given condition. Now by Corollary 4.5.7, we have

$$\text{either } cl(\mathcal{A}) = C(Y^*) \text{ or there is an } y_0 \in Y^* \text{ such that } cl(\mathcal{A}) = \{f \in C(Y^*) : f(y_0) = 0\}$$

Then immediately we have

$$\text{either } \overline{\mathcal{A}} = C_0(Y^*) \text{ or } \overline{\mathcal{A}} = \{f \in C_0(Y^*) : f(y_0) = 0\} \text{ for some } y_0 \in X.$$

where $\overline{\mathcal{A}}$ denotes the closure in the subspace since $C_0(Y^*,\mathbb{R})$ and $\{f \in C_0(Y^*,\mathbb{R}) : f(y_0) = 0\}$ denote the subspace of $C(Y^*)$ and $\{f \in C(Y^*,\mathbb{R}) : f(y_0) = 0\}$ respectively. Now since as a set, X is identical to Y^*, we have

$$\text{either } \overline{\mathcal{A}} = C_0(X,\mathbb{R}) \text{ or } \overline{\mathcal{A}} = \{f \in C_0(X,\mathbb{R}) : f(y_0) = 0)\}$$

Now we consider the second case. We denote X^* be the one-compactification of X. Since $f \in C_0(X,\mathbb{R})$ be extended to a function in $C(X^*,\mathbb{R}^*)$ in the following way,

$$\dot{f} : X^* \to \mathbb{R}^* \text{ defined by } \dot{f}(x) = f(x) \text{ on } X \text{ and } \dot{f}(\infty) = 0.$$

We can embed $\mathcal{A}$ into $C(X^*)$ as a subset. There is no doubt that $\mathcal{A}$ is still an sub algebra of $C(X^*)$ by the given condition. Now we need to check $\mathcal{A}$ separates points of X^*. We choose $x, y \in X^*$ with $x \neq y$ randomly. If $x, y \in X$, by the condition that $\mathcal{A}$ separates points, we can find that $\dot{f} \in \mathcal{A}$ such that $\dot{f}(x) \neq \dot{f}(y)$. If $x = \infty$, we can choose $\dot{f} \in \mathcal{A}$ such that $\dot{f}(y) \neq 0$ since we are in the second case. Then By Corollary 4.5.7, we have

$$\text{either } \overline{\mathcal{A}} = C(X^*) \text{ or there is an } y_0 \in X^* \text{ such that } \overline{\mathcal{A}} = \{f \in C(X^*) : f(y_0) = 0\}$$

Obviously there are bijections between the set $C(X^*)$ and $C_0(X,\mathbb{R})$ and the set $\{f \in C(X^*) : f(y_0) = 0\}$ and $\{f \in C_0(X,\mathbb{R}) : f(y_0) = 0\}$ and hence there are the identical topological spaces. So we proved that statement

$$\text{either } \overline{\mathcal{A}} = C_0(X,\mathbb{R}) \text{ or } \overline{\mathcal{A}} = \{f \in C_0(X,\mathbb{R}) : f(y_0) = 0\} \text{ for some } y_0 \in X^*.$$

□

Question 4.24: Let X and Y be compact Hausdorff spaces. and assume that $X \times Y$ is compact Prove that

$$\mathcal{A} = \Big\{ \sum_{i=1}^{n} g_i(x)h_i(y) : g_i \in C(X), h_i \in C(Y), n \in \mathbb{N} \Big\}$$

is dense in $C(X \times Y)$ with the uniform topology.

Proof. We want to use Corollary 4.5.7 to prove the result. The $X \times Y$ is compact is given by the condition. There is almost nothing to prove that $X \times Y$ is Hausdorff and this is given by the product topology. We also have the constant one function in $\mathcal{A}$ and this is because

$$1(x)1(y) \in \mathcal{A}$$

where $(X \ni x \mapsto 1 \in \mathbb{R}) \in C(X)$ and $(Y \ni y \mapsto 1 \in \mathbb{R}) \in C(Y)$. For any $(x_1, y_1), (x_2, y_2) \in X \times Y$ with $(x_1, y_1) \neq (x_2, y_2)$, we have $x_1 \neq x_2$ or $x_2 \neq y_2$ and without loss of generality, we assume that $x_1 \neq x_2$. Since $P(X)$ separates $C(X)$, we have $p_1, p_2 \in P(X)$ such that $p_1(x_1) \neq p_2(x_2)$. And hence $p_1(x_1)1(y_1) \neq p_2(x_2)1(y_2)$ where 1 denotes the constant function in $C(Y)$. To achieve the result

$$\overline{\mathcal{A}} = C(X)$$

,by Theorem 4.5.6, it only remains to show that $\mathcal{A}$ is a subalgebra. First we want to prove that $\mathcal{A}$ is a subalgebra. First, $\mathcal{A}$ is a vector subspace and there is no doubt it is closed under addition and scalar production. We need to check that $\mathcal{A}$ is closed under production. We choose $a, b \in \mathcal{A}$ randomly. By Definition of $\mathcal{A}$, we have

$$a(x, y) = \sum_{i=1}^{n} g_i(x)h_i(y)$$

for some $g_i \in C(X)$ and some $h_i \in C(Y)$ and $n \in \mathbb{N}$, also we have

$$b(x, y) = \sum_{i=1}^{m} g_i'(x)h_i'(y)$$

for some $g_i' \in C(X)$ and some $h_j' \in C(Y)$ and $m \in \mathbb{N}$. Then we have for $(x, y) \in X \times Y$

$$a(x,y)b(x,y) = \sum_{i=1}^{n} g_i(x)h_i(y) \sum_{j=1}^{m} g_j'(x)h_j'(y) = \sum_{i=1}^{n}\sum_{j=1}^{m} g_i(x)g_j'(x)h_i(y)h_j'(y) =$$
$$g_1(x)g_1'(y)h_1(x)h_1'(y) + \cdots + g_n(x)g_m'(x)h_m(y)h_m'(y)$$

and then there is no doubt that $ab \in \mathcal{A}$ since both $C(X)$ and $C(Y)$ are an algebra.

$\square$

Question 4.25: Let $g \in C([a,b])$ be strictly increasing, $a < b$ in $\mathbb{R}$. Prove that for any $f \in C([a,b])$ and for any $\epsilon > 0$, there are $a_0, a_1, \cdots, a_n \in \mathbb{R}$ such that

$$\max_{x\in[a,b]} |f(x) - (a_0 + a_1 g(x) + a_2 g^2(x) + \cdots + a_n g^n(x))| < \epsilon$$

Proof. We denote

$$\mathcal{B} := \{1, g, g^2, \cdots\}$$

and denote

$$\mathcal{A} = span(B).$$

We prove that $\mathcal{A}$ is a subalgebra of $C([0,1],\mathbb{R})$. From Definition of span, there is no doubt that $\mathcal{A}$ is a subspace of $C([a,b])$ over $\mathbb{R}$. Now it remains to prove that $\mathcal{A}$ is closed under multiplication. We choose $a, b \in \mathcal{A}$ randomly. By Definition of span, we can write

$$a := a_0 + a_1 g^1 + a_2 g^2 + a_3 g^2 + \cdots + a_n g^n \text{ and } b := b_0 + b_1 g^1 + b_2 g^2 + b_3 g^3 + \cdots + b_m g^m$$

for some $n, m \in \mathbb{N}$, $a_0, \cdots, a_n \in \mathbb{R}$ and some $b_0, \cdots, b_m \in \mathbb{R}$. Then we have

$$ab = a_0 b_0 + \cdots + a_n b_m g^{n+m} \in span(B)(= \mathcal{A})$$

since $(\mathbb{R}, \cdot)$ is a multiplicative group. But $a, b \in \mathcal{A}$ was chosen randomly. We proved that $\mathcal{A}$ is a subalgebra of $C([a,b])$ since it is itself an algebra over $\mathbb{R}$. Also the constant function $([0,1] \ni x \mapsto 1 \in \mathbb{R}) = 1 \cdot 1 \in \mathcal{A}$. For any $x, y \in [a,b]$ with $x \neq y$, by the monotonicity of g, $g(x) \neq g(y)$. Also $g \in \mathcal{B} \subset \mathcal{A}$. We proved that $\mathcal{A}$ separates points of $[a,b]$. Furthermore, since $[a,b]$ is compact Hausdorff space, by Stone-Weierstrass theorem, we have

$$\overline{\mathcal{A}} = C([a,b],\mathbb{R}).$$

We choose $f \in C([a,b])$ and $\epsilon > 0$ randomly. By Definition of dense and the norm space, we have

$$B(f,\epsilon) = \{h \in C([a,b]) : \|h - f\| < \epsilon\} \cap \mathcal{A} \neq \varnothing$$

which says that we can choose $h \in \mathcal{A}$ such that $\|h - f\| < \epsilon$. Then by Definition of span, we can write

$$h = a_0 + a_1 g + \cdots + a_n g^n$$

for some $n \in \mathbb{N}$ and some $a_0, \cdots, a_n \in \mathbb{R}$. Furthermore, by definition of sup norm and applying the extreme value theorem to $h - f$ on $[0,1]$ we have

$$\epsilon > \|h - f\| = \max_{x\in[a,b]} |h(x) - f(x)| = \max_{x\in[a,b]} |f(x) - (a_0 + a_1 g(x) + \cdots + a_n g^n(x))|$$

which says that we finish the proof since such $f \in C([a,b])$ and $\epsilon > 0$ were both chosen randomly. $\square$

Homework 5 — Math 5323
Due: Fridays, 25 Feb 2022, by 11:59 p.m. CDT

Question 5.1: Let X be a normed space over $\mathbb{R}$. Prove that the vector addition, $\alpha : X \times X \mapsto X; (x, y) \mapsto x + y$, the multiplication by a scalar, $\beta : \mathbb{R} \times X \to X; (c, x) \mapsto cx$, and the norm, $n : X \to \mathbb{R}; x \mapsto \|x\|$, are all continuous.

Proof. We choose $(a, b) \in X \times Y$ randomly. We choose $(x_i, y_i) \to (a, b)$ in the product space $X \times X$. We are required to prove that $\alpha(x_i, y_i) \to \alpha(a, b)$ in the space X, which is equivalent to prove that $\|\alpha(x_i, y_i) - \alpha(a, b)\| \to 0$ since the topology is induced by the norm. Actually we have by triangle inequalities

$$\|\alpha(x_i, y_i) - \alpha(a, b)\| = \|x_i + y_i - a - b\| \leq \|x_i - a\| + \|y_i - b\| \tag{1}$$

Since $(x_i, y_i) \to (a, b)$, by Definition of product spaces, we have $x_i \to a$ and $y_i \to b$ in X which implies that $\|x_i - a\| \to 0$ and $\|y_i - b\| \to 0$ due to the fact the topology on X is induced by the norm. Finally after pushing both sides of (1) into ∞, we proved α is continuous since $(a, b) \in X \times Y$ was chosen randomly.

We choose $(c, x) \in \mathbb{R} \times X$ randomly. We choose $(c_i, x_i) \to (c, x)$ in the product space $\mathbb{R} \times X$. We are required to prove that $\|\beta(c_i, x_i) - \beta(c, x)\| \to 0$ in the space X, which is equivalent to prove that $\|\beta(c_i, x_i) - \beta(c, x)\| \to 0$ since the topology is induced by the norm. Actually we have by triangle inequalities

$$\begin{aligned}\|\beta(c_i, x_i) - \beta(c, x)\| = \|c_i x_i - cx\| = \|c_i x_i - cx_i + cx_i - cx\| \leq \|c_i x_i - cx_i\| + \|cx_i - cx\| &= |c_i - c|\|x_i\| + |c|\|x_i - x\| \\ &\leq |c_i - c| \max_{i \in \mathbb{N}} \|x_i\| + |c|\|x_i - x\|\end{aligned} \tag{2}$$

where $\max_{i \in \mathbb{N}} \|x_i\| \in \mathbb{R}$ since the convergence sequence is bounded. Since $(c_i, x_i) \to (c, x)$, by Definition of product spaces, we have $c_i \to c$ in $\mathbb{R}$ and $x_i \to x$ in X which implies that $|c_i - c| \to 0$ and $\|x_i - x\| \to 0$ by Definition of normed spaces. Finally after pushing both sides of (2) into ∞, we proved that β is continuous since $(c, x) \in \mathbb{R} \times X$ was chosen randomly.

We choose $x \in X$ randomly. We choose $x_i \to x$ in the space X. Now we have

$$\lim_{i \to \infty} \Big| n(x_i) - n(x) \Big| = \lim_{i \to \infty} \Big| \|x_i\| - \|x\| \Big| \leq \lim_{i \to \infty} \|x_i - x\| = \| \lim_{i \to \infty} x_i - x\| = \|x - x\| = \|0\| = 0$$

where the equality is due to $\| \cdot \|$ is continuous, the second is by our choice of x_i and the last is due to properties of norm and the first inequality is due to triangles inequalities, which implies that $n(x_i) \to n(x)$ in $\mathbb{R}$ by the fact $(\mathbb{R}, | \cdot |)$ is a normed space.

□

Question 5.2: Let X be a normed space. Prove that the closure of a subspace Y is a subspace of X.

Proof. By Definition of closure, we have $\overline{Y} = \{y \in X : y = \lim_{i\to\infty} y_i, y_i \in Y \text{ for each } i \in \mathbb{N}\}$. Now by Definition of linear subspaces, it is closed to check it is closed under the addition and the scalar product. We choose $a, b \in \overline{Y}$ randomly. Then we can write $a = \lim_{i\to\infty} a_i$ and $b = \lim_{i\to\infty} b_i$ for $a_i, b_i \in Y$. By the product space, we have $(a_i, b_i) \to (a, b)$ and hence by Question 5.1, we have $a + b = \lim_{i\to\infty}(a_i + b_i)$. Since Y is a subspace of X, $a_i, b_i \in Y$ implies that $a_i + b_i \in Y$. So by Definition of $\overline{Y}$, we have $a + b \in \overline{Y}$. We choose $a \in \overline{Y}$ and $c \in \mathbb{R}$ randomly. Then we can write $a = \lim_{i\to\infty} a_i$ for $a_i \in Y$. Then by Definition of product spaces, we have $(c, a_i) \to (c, a)$ and hence by Question 5.1, we have $ca = \lim_{i\to\infty} ca_i$. Since Y is a subspace of X over $\mathbb{R}$, $c \in \mathbb{R}$ and $a_i \in Y$ implies that $ca_i \in Y$. So by Definition of $\overline{Y}$, we have $ca \in \overline{Y}$.

□

Question 5.4: Let X be a $n-$dimensional vector space over $\mathbb{R}$ and $\{e_1, \cdots, e_n\}$ be a basis for X. For any $x = a_1e_1 + \cdots + a_ne_n \in X$, defined

$$\|x\|_\infty = \max_i |a_i|$$

Prove the following statements.

(a.) $\|\cdot\|_\infty$ is a norm on X.

(b.) The map $\phi : (\mathbb{R}^n, \|\cdot\|) \to (X, \|\cdot\|_\infty)$ defined by

$$\phi((a_1, \cdots, a_n)) = a_1e_1 + \cdots + a_ne_n$$

is continuous, where $\|\cdot\|$ on $\mathbb{R}^n$ is the Euclidean norm.

(c.) The set $K = \{x \in X : \|x\|_\infty = 1\}$ is compact in $(X, \|\cdot\|_\infty)$. (Hint:$\phi^{-1}(K)$ is compact in $\mathbb{R}^n$.)

(d.) All norms on a finite dimensional vector space over $\mathbb{R}$ are equivalent.

Proof. (a) By Definition 5.1.1, we are going to check that

(i) There is no doubt that $\|\cdot\|_\infty : X \to [0, \infty)$ is a well-defined map.

(ii) $\|x\|_\infty = \max_i |a_i| = 0$ iff $a_i = 0$ for each i iff $x = 0$ by Definition of basis and Definition of max.

(iii) Now for any $x, y = b_1e_i + \cdots + b_ne_n \in X$, which implies that $x + y = (a_1 + b_1, \cdots, a_n + b_n)$, we have by triangle inequalities and Definition of max

$$\|x + y\|_\infty = \max_i |a_i + b_i| \leq \max_i |a_i| + \max_i |b_i| = \|x\|_\infty + \|y\|_\infty$$

(iv) For $c \in \mathbb{R}$, we have $cx = (ca_1, \cdots, ca_n)$ by Definition of linear spaces and Definition of basis. So $\|cx\|_\infty = \max_i |ca_i| = |c| \max_i |a_i| = |c|\|x\|_\infty$

(b) ϕ is a linear map since $\phi((a_1, \cdots, a_n) + (b_1, \cdots, b_n)) = a_1e_1 + \cdots + a_ne_n = \phi((a_1 + b_1, \cdots, a_n + b_n))$ and $c\phi((a_1, \cdots, a_n)) = (ca_1)e_1 + \cdots + (ca_n)e_n = \phi((ca_1, \cdots, ca_n))$ for any $(a_1, \cdots, a_n), (b_1, \cdots, b_n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Now by Proposition 5.2.2, it is enough to prove that ϕ is continuous at $(0, \cdots, 0) \in \mathbb{R}^n$. We choose $(a_1^{(i)}, \cdots, a_n^{(i)}) \to (0, \cdots, 0)$ in $\mathbb{R}^n$. Then we have

$$\|\phi((a_1^{(i)}, \cdots, a_n^{(i)})) - \phi((0, \cdots, 0))\|_\infty = \|a_1e_1^{(i)} + \cdots + a_ne_n^{(i)}\|_\infty = \max_k |a_k^{(i)}| \leq \sqrt{(a_1^{(i)})^2 + \cdots + (a_n^{(i)})^2} = \|(a_1^{(i)}, \cdots, a_n^{(i)}) - (0, \cdots, 0)\|$$

and we immediately proved the result after pushing both sides to ∞.

(c) Since ϕ is continuous, by the fact that continuous maps send compact sets to compact sets, it is enough to prove that

$$K = \phi(\phi^{-1}(K))$$

There is almost nothing to prove that $K \subset \phi(\phi^{-1}(K))$ and almost nothing to prove that $K \subset \phi(\phi^{-1}(K))$ either.

(d) We denote $n :=$ the dimensional of X for some $n \in \mathbb{N}$. We choose a set $\{e_1, \cdots, e_n\}$ to be a basis for X and we are only interested in the finite dimensional real vector spaces, but there is almost no extra work for the complex case. We denote $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ be norms on X. By the transitive property of equivalent norms, it is enough to prove that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_\infty)$ are equivalent. For simplicity, we denote $\|\cdot\| := \|\cdot\|_1$. We denote $C := \|e_1\| + \cdots + \|e_n\|$ for some $C \in (0, \infty)$. For any $x = x_1e_1 + \cdots + x_ne_n \in X$, by the triangle inequality of norms and properties of norms, we have

$$\|x\| = \|x_1e_1 + \cdots + x_ne_n\| \leq |x_1|\|e_1\| + \cdots + |x_n|\|e_n\| \leq \max_{1 \leq j \leq n} |x_j|\|e_1\| + \cdots + \max_{1 \leq j \leq n} |x_j|\|e_n\| = C\|x\|_\infty.$$

We denote the identity map $c : (X, \|\cdot\|_\infty) \to (X, \|\cdot\|); x \mapsto x$ and there is almost nothing to prove that c is linear. Then c is continuous by Proposition 5.2.2 and Definition 5.2.1. Since continuous maps send compact sets to compact sets, by (c.), we know that $c(K) := \{x \in X : \|x\|_\infty = 1\}$ is compact in $(X, \|\cdot\|)$. Now since $\|\cdot\| : K \to [0, \infty)$ is continuous by the extreme value theorem, we can denote $\|a\| := \min_{x \in K} \|x\|$ for some $a \in K$. Now for any $x \in X$, $\left\|\frac{x}{\|x\|_\infty}\right\|_\infty = \frac{\|x\|_\infty}{\|x\|_\infty} = 1$ implies that $\frac{x}{\|x\|_\infty} \in c(K)$ and hence $\|\frac{x}{\|x\|_\infty}\| = \frac{\|x\|}{\|x\|_\infty} \geq \|a\|$ implying that $\|x\| \geq \|a\|\|x\|_\infty$. So far we proved that for any $x \in X$,

$$\|a\|\|x\|_\infty \leq \|x\| \leq C\|x\|_\infty.$$

So by Definition 5.1.3, we finished the proof.

$\square$

Question 5.5: Let X, Y be a normed spaces and X be finite dimensional. Prove that any linear operator $T : X \to Y$ is bounded. (Hint:Show that T is continuous at 0. See exercise 5.4.)

Proof. We choose $T : X \to Y$ be a linear operator randomly. Since X, Y are both normed spaces, by Proposition 5.2.2, it is enough to prove that T is continuous at 0. Since X is a finite dimensional space, by Definition of finite dimensional space, we can choose a set of linear independent vectors $x_1, \cdots, x_n \in X$ such that $X = span\{x_1, \cdots, x_n\}$ where $n := dim(X)$ for some $n \in \mathbb{N}$. We randomly choose a sequence $y_j \in X$ such that $y_j \to 0$. Then for each $j \in \mathbb{N}$, we can write

$$y_j = \alpha_1^{(j)} x_1 + \cdots + \alpha_n^{(j)} x_n$$

for some $\alpha_1^{(j)}, \cdots, \alpha_n^{(j)} \in \mathbb{C}$. Then by properties of limit, we have

$$0 = \lim_{j\to\infty} y_j = \lim_{j\to\infty} (\alpha_1^{(j)} x_1 + \cdots + \alpha_n^{(j)} x_n) = \lim_{j\to\infty} (\alpha_1^{(j)} x_1) + \cdots + \lim_{j\to\infty} (\alpha_n^{(j)} x_n) = (\lim_{j\to\infty} \alpha_1^{(j)}) x_1 + \cdots + (\lim_{j\to\infty} \alpha_n^{(j)}) x_n \tag{3}$$

The last equality is true since for each $1 \le i \le n$, we have by properties of norm and definition of the vector addition for any $j \in \mathbb{N}$

$$\|\alpha_i^{(j)} x_i - (\lim_{j\to\infty} \alpha_i^{(j)}) x_i\|_X = |\alpha_i^{(j)} - \lim_{j\to\infty} \alpha_i^{(j)}| \|x_i\|_X$$

which immediately implies that after pushing j into ∞,

$$\alpha_i^{(j)} x_i \to (\lim_{j\to\infty} \alpha_i^{(j)}) x_i \text{ in } X.$$

From (3), by definition of a basis, we have that for each $1 \le i \le n$,

$$0 = \lim_{j\to\infty} \alpha_i^{(j)}.$$

Furthermore, by Definition of linear operators and properties of limit, we have

$$\begin{aligned}\lim_{j\to\infty} T(y_j) &= \lim_{j\to\infty} T(\alpha_1^{(j)} x_1 + \cdots + \alpha_n^{(j)} x_n) = (\lim_{j\to\infty} \alpha_1^{(j)} T(x_1)) + \cdots + (\lim_{j\to\infty} \alpha_n^{(j)} T(x_n)) \\ &= (\lim_{j\to\infty} \alpha_1^{(j)}) T(x_1) + \cdots + (\lim_{j\to\infty} \alpha_n^{(j)}) T(x_n) = 0 \cdot T(x_1) + \cdots + 0 \cdot T(x_n) = 0\end{aligned}$$

where the third equality is due to Lemma 0.1 ☐

Lemma 0.1. *Let $(X, \|\cdot\|)$ be a norm space over $\mathbb{C}$ and $x \in X$ and $\alpha_i \in \mathbb{C}$ be a convergent sequence with the limit $\alpha \in \mathbb{C}$. Prove that*

$$\alpha_i x \to \alpha x \textit{ in } X$$

Proof. Left for readers. ☐

Question 5.6: Prove that any finite dimensional real normed vector space is a Banach space. (Hint: Exercise 5.4.)

Proof. Firstly, we know that every real n-dimensional vector space is isomorphic to $(\mathbb{R}^n, \mathbb{R})$. So it is enough to prove that the normed space $(\mathbb{R}^n, \|\cdot\|)$ is a Banach space. By Question 5.4 (d), we know that the topology on a finite dimensional space is actually independent of equipped norms, so once we prove it is complete w.r.t a well-defined norm, it is complete w.r.t any well-defined norms. That is to say we can study the topology of a finite dimensional space by equipping it with a norm. By (a) of Question 5.4, it is enough to prove that $(\mathbb{R}^n, \|\cdot\|_\infty)$ is a Banach space. By Definition 5.1.2, it remains to prove the completeness. We randomly choose a Cauchy sequence $\{a_k\}_{k=1}^{\infty}$ in $(\mathbb{R}^n, \|\cdot\|_\infty)$. For each $k \in \mathbb{N}$, we denote $a_k = (a_k^{(1)}, \cdots, a_k^{(n)})$ for some $a_k^{(1)}, \cdots, a_k^{(n)} \in \mathbb{R}$ due to Definition of the product of sets. Now by Question 5.4(a), we have for each $1 \leq j \leq n$, for any $m, l \in \mathbb{N}$, we have

$$|a_m^{(j)} - a_l^{(j)}| \leq \max_{1 \leq p \leq n} |a_m^{(p)} - a_l^{(p)}| = \|a_m - a_l\|_\infty$$

where the last inequality is due to $a_m - a_l = (a_m^{(1)}, \cdots, a_m^{(n)}) - (a_l^{(1)}, \cdots, a_l^{(n)}) = (a_m^{(1)} - a_l^{(1)}, \cdots, a_m^{(n)} - a_l^{(n)})$, which says that for each $1 \leq j \leq n$, the real sequence $\{a_k^{(j)}\}_{k=1}^{\infty}$ is a Cauchy sequence in the normed space $(\mathbb{R}, |\cdot|)$. Then by its completeness, we have $a^{(j)} \in \mathbb{R}$ such that $a_k^{(j)} \to a^{(j)}$ in $\mathbb{R}$ for each $1 \leq j \leq n$. We denote $a = (a^{(1)}, \cdots, a^{(n)})$. There is no doubt that $a \in \mathbb{R}^n$. For less confusion, we denote $\|\cdot\| := \|\cdot\|_\infty$. Now it remains to prove that $\lim_{k\to\infty} \|a_k - a\| = 0$. For each $k \in \mathbb{N}$, we have

$$\|a_k - a\| = \max\{|a_k^{(p)} - a^{(p)}| : 1 \leq p \leq n\} = |a_k^{(p_0)} - a^{(p_0)}|$$

where we denote $p_0 :=$ the index which value is the maximum of the set $\{|a_k^{(p)} - a^{(p)}| : 1 \leq p \leq n\}$ and the second equality is due to the definition of max and the set is finite, which immediately implies that after pushing k into ∞,

$$\|a_k - a\| \to 0$$

since $1 \leq p_0 \leq n$. □

Remark 0.2. *The argument can be rehearsed for a finite-dimensional complex normed space.*

Homework 6 — Math 5323
Due: Fridays, 4 Mar 2022, by 11:59 p.m. CDT

Question 5.7: Let X be a Banach space.

(a) Prove that for any $R \in L(X, X)$ with $\|R\| < 1$, $\sum_{i=0}^{\infty} R^i$ converges in $L(X, X)$ and $I - R$ is invertible with $(I - R)^{-1} = \sum_{i=0}^{\infty} R^i$.

(b) Prove that the set of all invertible operators in $L(X, X)$ is open by showing that if T is invertible and

$$\|S - T\| < \tfrac{1}{\|T^{-1}\|},$$

then S is invertible. (Hint: $ST^{-1} = I - (I - ST^{-1})$ and the given condition implies that $\|I - ST^{-1}\| < 1$.)

Proof. (a) We choose $R \in L(X, X)$ with $\|R\| < 1$ randomly. By Definition, we are required to prove that $\{\sum_{i=0}^{n} R^i\}_{n=1}^{\infty}$ is converges in $L(X, X)$. Since X is Banach space, by Definition 5.1.2, we know that X is complete. Then by Proposition 5.2.5., we know that $L(X, X)$ is complete. Now by Definition of completeness, it is enough to prove that $\{\sum\limits_{i=0}^{n} R^i\}_{n=1}^{\infty}$ is Cauchy. We choose $\epsilon > 0$ randomly. We are required to prove that there exists $N \in \mathbb{N}$ such that

$$A_{m,n} := \Big\| \sum_{i=1}^{n} R^i - \sum_{i=1}^{m} R^i \Big\| < \epsilon \tag{1}$$

for any $n \geq m \geq N$. But actually we have by triangle inequalities,

$$A_{m,n} = \Big\| \sum_{i=m+1}^{n} R^i \Big\| \leq \sum_{i=m+1}^{n} \|R^i\| \underbrace{\leq}_{*} \sum_{i=m+1}^{n} \|R\|^i = \Big| \sum_{i=m+1}^{n} \|R\|^i \Big| = \Big| \sum_{i=1}^{n} \|R\|^i - \sum_{i=1}^{m} \|R\|^i \Big| =: a_{m,n}. \tag{2}$$

and we have that there exists $N \in \mathbb{N}$

$$a_{m,n} < \epsilon \tag{3}$$

for any $n \geq m \geq N$ by Definition of completeness since $\sum\limits_{i=1}^{\infty} \|R\|^i$ where $\|R\| < 1$ is a geometric series and $\mathbb{R}$ is complete. So combing (2) and (3), we proved (2) immediately. To complete the whole proof, it only remains to prove $(*)$. We can prove it by induction on i and the inductive step can be finished quickly by Remark 5.1 where $R^i := \underbrace{R \circ \cdots \circ R}_{i \text{ terms}}$. By Definition of groups, we are required to prove that

$$(I - R) \circ A = I \tag{4}$$
$$A \circ (I - R) = I \tag{5}$$

where we denote $A := \sum\limits_{i=0}^{\infty} R^i$ for some $A \in L(X, X)$. But actually we have by distributive law of composition of maps and $A \in L(X, X)$

$$(I - R) \circ A = (I - R) \circ (I + R + R^2 + R^3 + \cdots) = (I + R + R^2 + R^3 + \cdots) - (R + R^2 + R^3 + \cdots) = A - (A - I) = I$$

which says that we proved (4) and similarly we can prove (5).

(b) We denote $\mathcal{B} := \{T \in L(X, X) : T \text{ is invertible }\}$ and we choose $T \in B$ randomly. Since open balls form a basis for the topology $L(X, Y)$, by Definition of basis, we are required to find $R > 0$ such that for any $S \in L(X, Y)$ with $\|S - T\| < R$, S is invertible. By Remark 5.2 and Definition $L(X, X)$, we have $0 < \|T \circ T^{-1}\| \leq \|T\| \|T^{-1}\|$ which implies that $\|T^{-1}\| > 0$. Now we prove that $\frac{1}{\|T^{-1}\|}$ is a desired R. We choose $S \in L(X, X)$ with $\|S - T\| < \frac{1}{\|T^{-1}\|}$ randomly. By Remark 5.2, we have $\|I - ST^{-1}\| = \|T^{-1}(S - T)\| \leq \|T^{-1}\| \|S - T\| < 1$ and hence by (a), $A := (I - (I - ST^{-1}))^{-1} = (ST^{-1})^{-1}$ exists in $L(X, X)$. By Definition of invertible maps, we have

$$A \circ (S \circ T^{-1}) = I \text{ and } (S \circ T^{-1}) \circ A = I$$

which implies that

$$(T^{-1} \circ A) \circ S = T^{-1} \circ (A \circ S \circ T^{-1}) \circ T = I \text{ and } S \circ (T^{-1} \circ A) = I$$

where we conjugate the first equation by T and here we note that the group $(L(X,Y), \circ)$ is not commutative, which implies that $S^{-1} = T^{-1} \circ A$ exists in $L(X,X)$. Since such S was chosen randomly, we proved that $\mathcal{B}$ is open in $L(X,X)$.

$\square$

Question 5.8: Let X be a normed vector space and M be a proper closed subspace of X. Prove the following statements

a. $$\|x + M\| = \inf\{\|x + y\| : y \in M\}$$
is a norm on X/M.

b. For any $\epsilon > 0$, there exists an $x \in X$ such that $\|x\| = 1$ and $\|x + M\| \geq 1 - \epsilon$.

c. The projection $\pi : X \to X/M$ defined by $\pi(x) = x + M$ has norm 1.

d. X/M is complete if X is complete.

Proof. (a) We confirm that

(i) There is no doubt that the map $X/M \to [0, \infty); x + M \mapsto \inf\{\|x + y\| : y \in M\}$ is well defined since by Definition of inf, it cannot achieve ∞.

(ii) Now for any $x \in M$, we have $-x \in M$ by Definition of subspaces, and hence $\|x + M\| = 0$ by Definition of inf and Definition 5.1.1 (a) since $(X, \|\cdot\|)$ is a normed space. Also if $\|x + M\| = 0$, then by Definition of inf and Definition of normed spaces, we have as sequence $-y_k \in M$ such that $-y_k \to x$ in X. By Definition of closeness, we have $x \in M$ which immediately implies that $x + M = 0 \in X/M$ by Definition of quotient spaces.

(iii) We choose $x + M, y + M \in X/M$ randomly. By Definition of triangle, we are required to prove that
$$\|[x] + [y]\| \leq \|[x]\| + \|[y]\|.$$
By Definition of inf, we can choose a sequence $a_k \in M$ such that $\|[x]\| = \lim_{k\to\infty} \|x + a_k\|$ and similarly we can choose a sequence $b_k \in M$ such that $\|[y]\| = \lim_{k\to\infty} \|y + b_k\|$. Then we have
$$\|[x]\| + \|[y]\| = \lim_{k\to\infty} \|x + a_k\| + \lim_{k\to\infty} \|y + b_k\| \geq \lim_{k\to\infty} \|(x + y) + (a_k + b_k)\| \geq \|[x + y]\|$$
where the first inequality is due to triangle inequalities and the second inequality is due to Definition stated in the question where $a_k, b_k \in M$ implies that $a_k + b_k \in M$ by Definition of subspaces

(iv) We choose $c \in \mathbb{R}$ and $[x] \in X/M$ randomly. We are required to prove that
$$|c|\|[x]\| = \|[cx]\| \tag{6}$$
(6) holds obviously for $c = 0$. We consider the other case. For any $z \in M$, by Definition of inf, we have, since $\frac{z}{c} \in M$ by Definition of subspaces,
$$\|cx + z\| = |c|\|x + \tfrac{z}{c}\| \geq |c|[x].$$
This immediately implies that
$$\|[cx]\| = \inf\{\|cx + z\| : z \in M\} \geq |c|\|[x]\|. \tag{7}$$
For any $z \in M$, by Definition of inf, since $cz \in M$ by Definition of subspaces, we have
$$\inf\{\|cx + z\| : z \in M\} \leq |c|\|x + z\| = \|cx + cz\| \tag{8}$$
which immediately implies that
$$\frac{1}{|c|} \inf\{\|cx + z\| : z \in M\} \leq \|x + z\|. \tag{9}$$
This immediately implies that
$$\frac{1}{|c|}\|[cx]\| \leq \|[x]\| \tag{10}$$
which implies that
$$\|[cx]\| \leq |c|\|[x]\| \tag{11}$$
Combing (11) and (7), we proved (6) immediately.

(b) We choose $\epsilon > 0$ randomly. If $\epsilon \geq 1$, there is nothing to prove since the norm is non negative function and for any $z \in X \setminus \{0\}$ we have $\|\frac{z}{\|z\|}\| = \frac{\|z\|}{\|z\|} = 1$ by properties of norms. So we can consider the case $0 \leq \epsilon < 1$. We are not interested in the trivial case $X/M = \{0\}$. Now we choose $z_1 \in X \setminus M$. Then $\frac{\|z_1+M\|}{1-\epsilon} \geq \|z_1 + M\|$ in this case and by Definition of inf, we can choose $y \in M$ such that

$$\frac{\|z_1 + M\|}{1-\epsilon} \geq \|z_1 + y\| \geq \|z_1 + M\| \tag{12}$$

Now we denote $z := \frac{z_1+y}{\|z_1+y\|}$ where $z_1 \notin M$ implies that $z_1 + y \neq 0$ and consequently $\|z_1 + y\| \neq 0$. Then we have

$$\|z + M\| = \|[z]\| = \|[\tfrac{z_1+y}{\|z_1+y\|}]\| = \tfrac{1}{\|z_1+y\|}\|[z_1 + y]\| = \tfrac{1}{\|z_1+y\|}\|[z_1]\| \geq 1 - \epsilon$$

where the second equality is by Definition of z_1, the third is due to Definition of norms, the forth is due to $y \in M$ and the last inequality is due to (12).

(c) There is no doubt that the canonical map π is linear from X to X/M and by Definition of $L(X, X/M)$, we are required to prove

$$\sup\{\|z + M\| : z \in X \text{ and } \|z\| = 1\} = 1 \tag{13}$$

By (c), we have that for any $\epsilon > 0$, there exists $z \in M$ such that

$$\sup\{\|z + M\| : z \in X \text{ and } \|z\| = 1\} \geq \|z + M\| \geq 1 - \epsilon$$

by Definition of sup. This immediately implies that

$$\sup\{\|z + M\| : z \in X \text{ and } \|z\| = 1\} \geq \|z + M\| \geq 1 \tag{14}$$

For any $z \in X$ with $\|z\| = 1$, we have

$$\|z + M\| \leq \|z + 0\| = \|z\| = 1$$

by Definition of norms with $0 \in M$. By Definition of sup, this implies that

$$\sup\{\|z + M\| : z \in X \text{ and } \|z\| = 1\} \leq 1 \tag{15}$$

Combing (14) and (15), we proved (13).

(d) We randomly choose an absolutely convergent series $\sum_{n=1}^{\infty} a_n + M$ in X/M. Now by Theorem 5.1.5, it is equivalent to prove that $\sum_{n=1}^{\infty} a_n + M$ converges. By Definition, we have

$$\sum_{n=1}^{\infty} \|a_n + M\| < \infty. \tag{16}$$

Now for each $n \in \mathbb{N}$, by Definition of quotient norms, we have $b_n \in M$ such that

$$\|a_n - b_n\| \leq \|a_n + M\| + \frac{1}{2^n}. \tag{17}$$

Combing (16) and (17), we have

$$\sum_{n=1}^{\infty} \|a_n - b_n\| \leq \sum_{n=1}^{\infty} \|a_n + M\| + \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

which says that the series $\sum_{n=1}^{\infty}(a_n - b_n)$ is absolutely converges. Since X is a Banach space, by Theorem 5.1.5, we have the series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges in X. We denote $x := \sum_{n=1}^{\infty}(a_n - b_n)$ for some $x \in X$. Now it remains to prove that $\sum_{k=1}^{n} a_k + M \to x + M$ in X/M. But actually we have for each $k \in \mathbb{N}$

$$\|\sum_{k=1}^{n}(a_k + M) - (x + M)\| = \|\sum_{k=1}^{n}(a_k + M) - (x + M) - \sum_{k=1}^{n}(b_k + M)\| \tag{18}$$

$$= \|(\sum_{k=1}^{n}(a_k + b_k) - x) + M\| \tag{19}$$

$$\leq \|\sum_{k=1}^{n}(a_k - b_k) - x\| \tag{20}$$

where (18) is due to $b_k \in M$ for each k, (19) is due to Definition of addition for quotient spaces and (19) is due to Definition of norms for quotient spaces, which immediately implies that after pushing n into ∞,

$$\lim_{n\to\infty} \|\sum_{k=1}^{n}(a_k + M) - (x + M)\| \leq \lim_{n\to\infty} \|\sum_{k=1}^{n}(a_k - b_k) - x\| = 0$$

which says that

$$\sum_{k=1}^{n} a_k + M \to x + M \text{ in } X/M.$$

□

Lemma 0.1. *Let X, Y be normed spaces. Let $L(X, Y)$ denoted the normed space collecting bounded linear maps from X to Y. Let $x \in X$ and $T \in L(X, Y)$. Prove that $\|L\|\|x\| \geq \|L(x)\|$*

Proof. For $\|x\| = 0$, we have nothing to prove. We consider the case $\|x\| > 0$. We have

$$\|L\| \geq \|L(\tfrac{x}{\|x\|})\| = \tfrac{1}{\|x\|}\|L(x)\|$$

where the first is by Definition of operator norms since $\|\frac{x}{\|x\|}\| = 1$ and the second is by Definition of norms, which implies that

$$\|L\|\|x\| \geq \|L(x)\|$$

□

Question 5.9: Let X and Y be normed spaces and $T \in L(X,Y)$. Prove that

$$N(T) := \{x \in X : T(x) = 0\}$$

is a closed subspace of X.

Proof. By Definition of inverse images, we have $N(T) = T^{-1}(\{0\})$=ker(T). From basic algebra knowledge, there is no doubt that $N(T)$ is a subspace of X. Now it remains to prove that $N(T)$ is closed in X. By Proposition 5.2.2, we know that T is continuous. By Remark 4.1, it is enough to prove that $\{0\}$ is closed in Y. There is no doubt that $\{0\}$ is compact. By Proposition 4.3.7, it is enough to prove that Y is Hausdorff. We choose $x, y \in Y$ with $x \neq y$ randomly. By Definition of basis, we know that $B(x, \frac{r}{2})$ and $B(y, \frac{r}{2})$ open in Y where $r := \|x - y\| > 0$. Obviously $B(x,r) \cap B(y,r) \neq \varnothing$. Since such x, y was chosen randomly, by Definition 4.1.4(c), we proved Y is Hausdorff and hence we finished the proof. $\square$

Question 5.11: Prove that a linear functional f on a normed space X is bounded if and only if $f^{-1}(\{0\})$ is closed. (Hint: If $|f(x_n)| \to \infty$ for a sequence of unit vectors $\{x_n\}$, then $y_n = \frac{x_n}{f(x_n)} \to 0$, but $f(y_n) \nrightarrow 0$.)

Proof. We denote $f : X \to \mathbb{R}$ be a linear functional. First we want to prove that if f is bounded, then $f^{-1}(\{0\})$ is closed. By Proposition 5.2.2, we know T is continuous. By Remark 4.1, it is enough to prove that $\{0\}$ is closed in $\mathbb{R}$. Since $\mathbb{R}$ is Hausdorff and $\{0\}$ is compact, by Proposition 4.3.7, we immediately proved that $\{0\}$ is closed in $\mathbb{R}$ and hence we finished the implication.
Second we want to prove that if $f^{-1}(\{0\})$ is closed, then f is bounded. Equivalently we prove its contraposition, i.e., if f is unbounded, then $f^{-1}(\{0\})$ is not closed. By Definition 5.2.3, we know that $\sup\limits_{\|x\|=1} |f(x)| = \infty$. By Definition of sup, we can choose a sequence $x_k \in X$ with $\|x_k\| = 1$ such that $|f(x_k)| \to \infty$. We choose $e \in X$ such that $f(e) = 1$. We can always do it, since we are not interested in a trivial map and we can choose $a \in X$ such that $f(a) \neq 0$ and consider $\frac{a}{f(a)} \in X$. Now we consider the sequence $y_k := e - \frac{x_k}{f(x_k)}$ in X. We have $f(y_k) = f\Big(e - \frac{x_k}{f(x_k)}\Big) = f(e) - \frac{f(x_k)}{f(x_k)} = 0$ by linearity of f. So we have $y_k \in f^{-1}(\{0\})$ for each k. We have by Definition of norms and our choice of x_k for each $k \in \mathbb{N}$

$$\|y_k - e\| = \left\| \frac{x_k}{f(x_k)} \right\| = \frac{1}{|f(x_k)|} \|x_k\| = \frac{1}{|f(x_k)|} \tag{21}$$

which implies that by our choice of x_k, after pushing both sides of (16) into ∞, we proved that $y_k \to 0$ in X due to the topology on X is induced by norms. But we have $f(y_k) = f(\frac{x_k}{f(x_k)}) = \frac{f(x_k)}{f(x_k)} = 1$ by linearity of f. So we find a sequence $y_k \in f^{-1}(\{0\})$ such that $y_k \to e$ with $e \notin f^{-1}(\{0\})$. By Definition of closeness using sequences, we proved that $f^{-1}(\{0\})$ is not closed. $\square$

Lemma 0.2. *Let X be a normed space and $f : X \to \mathbb{R}$ be a linear map. Assume that $f^{-1}(\{0\})$ is closed. Prove that f is bounded.*

Proof. Idea: the composition of continuous maps is continuous and the natural projection of quotient spaces is continuous. $\square$

Question 5.12: Let X be a normed vector space.

(a) Prove that if M is a closed subspace and $x \notin M$,then $M + \mathbb{C}x$ is closed in X. (Hint: Theorem 5.2.10a).)

(b) Prove that every finite-dimensional subspace of X is closed.

Proof. (a) We note that $M + \mathbb{C}x = \{z + cx : z \in M, c \in \mathbb{C}\}$. We randomly choose a sequence $z_k + c_k x \in M + \mathbb{C}x$ such that $z_k + c_k x \to a$ in X. By Definition of closeness, it is enough to show that $a \in M + \mathbb{C}x$. Since X is a normed space with M a closed subspace, By Theorem 5.2.10(a), we can choose $f \in X^*$ such that $f(x) \neq 0$, $f\Big|_M = 0$ and $\|f\| = 1$. Then we have for each $k \in \mathbb{N}$ by Linearity of f , $f(z_k) = 0$ and Lemma 0.1

$$\Big|c_k f(x) - f(a)\Big| = \Big|f(c_k x - a) + f(z_k)\Big| = \Big|f(z_k + c_k x - a)\Big| \leq \|f\|\|z_k + c_k x - a\| = \|z_k + c_k x - a\|$$

which immediately implies $c_k f(x) \to f(a)$ in $\mathbb{R}$ after pushing both sides into ∞ by our choice of $z_k + c_k x$, which implies that $c_k \to \frac{f(a)}{f(x)}$ by properties of limits with $f(x) \neq 0$. Now since $a = a - \frac{f(a)}{f(x)}x + \frac{f(a)}{f(x)}x$, to finish the proof it remains to show $a - \frac{f(a)}{f(x)}x \in M$. Since M is closed and $z_k \in M$, it is enough to show that $z_k \to a - \frac{f(a)}{f(x)}x$ in X. Now we have by triangle inequalities and properties of norms, for each $k \in \mathbb{N}$

$$\left\|z_k - (a - \frac{f(a)}{f(x)}x)\right\| = \left\|z_k + c_k x - a + \frac{f(a)}{f(x)}x - c_k x\right\| \leq \|z_k + c_k x - a\| + \left|c_k - \frac{f(a)}{f(x)}\right|\|x\|$$

which immediately implies that $z_k \to a - \frac{f(a)}{f(x)}x$ after pushing both sides into ∞ by $z_k + c_k x \to a$ in X and $c_k \to \frac{f(a)}{f(x)}$ in $\mathbb{R}$, and $\|x\| < \infty$. So we finished the proof.

(b) We denote $M \subset X$ be a finite-dimensional subspace randomly. We denote $n := dim(M)$ for some $n \in \mathbb{N}$ by Definition of subspaces to be finite-dimensional since M is finite-dimensional. We want to prove that M is closed by induction on n. For the base step $n = 1$, by Definition of basis, we can write $M = span\{x\} = \mathbb{C}x$ (over $\mathbb{C}$) for some $x \in M \setminus \{0\}$. Then we have $M = \{0\} + \mathbb{C}x$. There is no doubt that the trivial subspace is closed in the normed space X. Also $x \neq 0$. Then by (a), M is closed. For the inductive step, we assume the statement:

every finite dimensional subspace of X is closed

is true when this subspace is of dimension n and we are required to prove that this statement holds when this subspace is of dimension $n + 1$. By Definition of basis and the knowledge of linear algebra, we can write $M = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_{n+1} = N + \mathbb{C}x_{n+1}$ for some basis $\{x_1, x_2, \cdots, x_{n+1}\}$ where we denote $N := \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$. Since M is of dimension $n + 1$, $x_{n+1} \in N$ and by the knowledge of linear algebra, N is finite-dimensional subspace of X of dimension n. Then by the inductive hypothesis, we know that N is closed. Furthermore since $x_{n+1} \notin N$, by (a), we have $M = N + \mathbb{C}x_{n+1}$ is closed. So we proved this statement holds for $n+1$ and hence we finished the inductive step and we proved the result by the induction method.

$\square$

Homework 7 — Math 5323

Due: Fridays, 11 Mar 2022, by 11:59 p.m. CDT

Question 5.13: Let X be an infinite -dimensional normed vector space.

a. Prove that there is a sequence of unit vectors $\{x_j\}$ such that $\|x_j - x_k\| \geq \frac{1}{2}$ fo $j \neq k$. (See Exercise 5.8 b.)

b. Prove that X is not locally compact by showing that $B_\epsilon = \{x \in X : \|x\| \leq \epsilon\}$ is not compact for any $\epsilon > 0$.(Hint: Consider $\{\epsilon x_j\}$ for the $\{x_j\}$ in a.)

Proof. a. First after normalizing, we can choose a unit vector $x_1 \in X$. There is no doubt that $span\{x_1\}$ is a proper closed subspace of X by Question 5.12 b.) since X is an infinite-dimensional normed space. Applying Exercise 5.8 b) with $span\{x_1\}$ and $\epsilon = \frac{1}{2}$, we can choose a unit vector x_2 such that $\|x_2 + span\{x_1\}\| \geq \frac{1}{2}$. Then by Definition of norms and Definition of span, we have $\|x_2 - x_1\| \geq \|x_2 + span\{x_1\}\| \geq \frac{1}{2}$. Inductively we can choose n such unit vectors $x_n \in X$. There is no doubt that $span\{x_1, \cdots, x_n\}$ is a proper closed subspace of X by Question 5.12 b.) since X is an infinite-dimensional normed space. Applying 5.8 b) with $span\{x_1, \cdots, x_n\}$ and $\epsilon = \frac{1}{2}$, we can choose a unit vector $x_{n+1} \in X$ such that $\|x_{n+1} + span\{x_1, \cdots, x_n\}\| \geq \frac{1}{2}$. For each $i \in \{1, \cdots, n\}$, by Definition of inf and Definition of span, we have $\|x_{n+1} - x_i\| \geq \|x_{n+1} + span\{x_1, \cdots, x_n\}\| \geq \frac{1}{2}$. By our construction of such sequence $x_n \in M$ of unit vectors , there is no doubt that for any $j \neq k$ $\|x_j - x_k\| \geq \frac{1}{2}$. We use constructive proof to finish this part.

b. We choose $\epsilon > 0$ randomly. By Definition 4.4.1 and Definition of basis, and the topology on X is induced by a norm, it is enough to prove that $B_\epsilon := \{x \in X : \|x\| \leq \epsilon\}(= \overline{B(0, \epsilon)})$ is not compact. Since any normed space is a metric space and in a metric space, a space is compact if and only if it is sequentially compact, it is equivalent to find a sequence $y_j \in B_\epsilon$ without a convergent sequence. Now by (a), we can choose a sequence $x_j \in X$ of unit vectors such that $\|x_j - x_k\| \geq \frac{1}{2}$ for $j \neq k$. For each $j \in \mathbb{N}$, we denote $y_j := \epsilon x_j$. Now it is enough to prove that such sequence $y_j \in X$ is the one in B_ϵ which is lack of a convergent sequence. There is no doubt that $y_j \in B_\epsilon$ by our choice of x_j since $\|y_j\| = \|\epsilon x_j\| = |\epsilon| \|x_j\| = |\epsilon|$. Now if it has a convergent sequence y_{j_k} then it must be a Cauchy sequence, then we can find $N \in \mathbb{N}$ such that by Properties of norms with $\epsilon > 0$

$$\|y_{j_N} - y_{j_{N+1}}\| = \|\epsilon x_{j_N} - \epsilon x_{j_{N+1}}\| = |\epsilon| \|x_{j_N} - x_{j_{N+1}}\| = \epsilon \|x_{j_N} - x_{j_{N+1}}\| < \tfrac{\epsilon}{2}$$

which implies that

$$\|x_{j_N} - x_{j_{N+1}}\| < \tfrac{1}{2}$$

which obviously contradicts our choice of x_j since $j_N \neq j_{N+1}$. So such sequence $y_j \in B_\epsilon$ has no convergent sequences.

□

Question 5.14: Let M be an finite-dimensional subspace of a normed space X. Prove that there is a closed subspace N such that $X = M \bigoplus N$,i.e., $X = M + N$ and $M \cap N = \{0\}$. (Hint: Theorem 5.2.10 (a))

Proof. We denote $n := dim(M)$ for some $n \in \mathbb{N}$ since M is of finite-dimensional. We denote $\{e_1, \cdots, e_n\} \subset M$ be a basis for M. For each $j = 1, \cdots, n$, we denote $A_j = span(\{e_1, \cdots, e_n\} \setminus \{e_j\})$. For each j, there is no doubt that A_j is a closed subspace of X by Question 5.12(b). Then for each j, By Theorem 5.2.10 with A_j and $e_j \notin A_j$, we have $f_j \in X^*$ with $f_j\big|_{A_j} = 0$ and $f_j(e_j) \neq 0$. We denote $N := \bigcap\limits_{j=1}^{n} ker(f_j)$. Now we prove that $X = M + N$ and $M \cap N = \{0\}$. We choose $x \in M \cap N$. Since $x \in M$, we can write $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$ by Definition of basis. Now $x \in N$ implies that for each j, $0 = f_j(x) = f_j(\alpha_1 e_1 + \cdots + \alpha_n e_n) = \alpha_1 f_j(e_1) + \cdots + \alpha_n f_j(e_n) = \alpha_j f_j(e_j)$ which implies that $\alpha_j = 0$ by our choice of f_j. So $x = 0$. We proved that $M \cap N = \{0\}$. There is no doubt that $M + N \subset X$. It remains to prove that $M + N \supset X$. We choose $z \in X$ randomly. For each $j = 1, \cdots, n$, we denote $\alpha_j := \frac{f_j(z)}{f_j(e_j)}$. Now since

$$z = (z - \alpha_1 e_1 - \cdots - \alpha_n e_n) + (\alpha_1 e_1 + \cdots + \alpha_n e_n)$$

, to prove that $z \in M + N$, it is equivalent to prove that $z - \alpha_1 e_1 - \cdots - \alpha_n e_n \in N$. For each j, by linearity of f_j and our choice of f_j and Definition of α_j

$$f_j(z - \alpha_1 e_1 - \cdots - \alpha_n e_n) = f_j(z) - \alpha_1 f_j(e_1) - \cdots - \alpha_n f_j(e_n) = f_j(z) - \alpha_j f_j(e_j) = 0,$$

which implies that $z - \alpha_1 e_1 - \cdots - \alpha_n e_n \in ker(f_j)$. This immediately implies that $z - \alpha_1 e_1 - \cdots - \alpha_n e_n \in N$. by Definition N. Since $z \in X$ was chosen randomly, we finished the whole proof.

□

Question 5.16: Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X, and both $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach spaces. Assume there is a $c > 0$ such that $\|\cdot\|_1 \le c\|\cdot\|_2$. Prove the norms are equivalent. (Hint: Corollary 5.2.17.)

Proof. By Definition 5.1.3, we are required to find $c_1, c_2 > 0$ such that

$$c_1\|x\|_1 \le \|x\|_2 \le c_2\|x\|_1 \text{ for any } x \in X$$

Now $\|\cdot\|_1 \le c\|\cdot\|_2$ implies that $\frac{1}{c}\|\cdot\|_1 \le \|\cdot\|_2$ where $c > 0$.So we find $c_1 := \frac{1}{c}$. Now it remains to find such $c_2 > 0$. We consider the identity map $T : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1); x \mapsto x$. There is no doubt that T is a bijective linear map. Since $\|T(\cdot)\|_1 = \|\cdot\|_1 \le c\|\cdot\|_2$, by Definition 5.2.1, $T \in L(X, X)$. Since $(X, \|\cdot\|_2)$ and $(X, \|\cdot\|_1)$ are both Banach spaces, by Corollary 5.2.17, we know T is an isomorphism. By Definition $T^{-1} : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is bounded. By Definition 5.2.1, we can choose $c_2 > 0$ such that

$$\|\cdot\|_2 = \|T^{-1}(\cdot)\|_2 \le c_2\|\cdot\|_1$$

where in Definition of inverse functions, $T^{-1}(x) = x$ for any $x \in X$. We found such $c_2 > 0$ and hence we finished the proof. $\square$

Question 5.17: Let X and Y be Banach spaces, $T \in L(X,Y)$, $N(T) = \{x \in X : T(x) = 0\}$, and $M = range(T)$. Prove that $X/N(T)$ and M are isomorphic if and only if M is closed. (See Exercises 5.8 and 5.9.)

Proof. First we prove that if $X/N(T)$ and M are isomorphic, then M is closed. We choose a sequence $y_k \in M$ such that $y_k \to y$ in Y. By Definition of closeness, we are required to prove that $y \in M$. By Definition of the range of a maps, we can write $y_k = T(x_k)$ for some $x_k \in X$ for each k. Now by Definition of closeness, it remains to prove that $y \in M$. Since $X/N(T)$ and M are isomorphic, by Definition 5.2.6 , we know that $S^{-1} : M \to X/N(T); y \mapsto x + N(T)$ where $T(x) = y$ is a bounded linear. Then by Definition 5.2.1, we have some $c > 0$ such that for any $k, l \in \mathbb{N}$,

$$\|(x_k + N(T)) - (x_l + N(T))\| \leq c\|y_k - y_l\| \tag{1}$$

Since y_k converges in Y, y_k is Cauchy in Y. Then by (1), $x_k + N(T)$ is Cauchy in $X/N(T)$. Since X is Banach space and $N(T)$ is a closed subspace of X by Question 5.9., $X/N(T)$ is Banach by Question 5.8(d). So by Definition 5.12., $x_k + N(T) \to x + N(T)$ for some $x \in X$. Since $S : X/N(T) \to M; x + N(T) \mapsto T(x)$ is a bounded linear map, by Definition 5.2.1, we have some $d > 0$ such that for each $k \in \mathbb{N}$,

$$\|T(x) - y_k\| = \|T(x) - T(x_k)\| \leq d\|(x + N(T)) - (x_k + N(T))\|.$$

Then by $x_k + N(T) \to x + N(T)$ in $X/N(T)$, we have $y_k \to T(x)$ in Y after pushing both sides into ∞. We know that normed space must be Hausdorff and the limit must be unique in Hausdorff. So $y_k \to y$ and $y_k \to T(x)$ implies that $y = T(x)$, which implies that $y \in M = range(T)$ immediately by Definition of the range of a map.

Second we want to prove that if M is closed , then $X/N(T)$ and M are isomorphic. By Definition 5.2.6, it is enough to prove that

$$S : X/N(T) \to M; x + N(T) \mapsto T(x)$$

is a well-defined bijective map, $S \in L(X/N(T), M)$ and $S^{-1} \in L(M, X/N(T))$. S is well-defined map since for any $x_1, x_2 \in N$, $x_1 + N(T) = x_2 + N(T)$ implies that $T(x_1) - T(x_2) = T(x_1 - x_2) = 0$ by Definition of $L(X,Y)$ and Definition of $N(T)$ since $T \in L(X,Y)$. S is linear since for $x, y \in X$ and $\alpha \in \mathbb{C}$, by Definition of quotient spaces, Definition of $L(X,Y)$ and Definition of S, we have

$$S(\alpha(x + N(T)) + (y + N(T))) = S((\alpha x + y) + N(T)) = T(\alpha x + y) = \alpha T(x) + T(y) = \alpha S(x + N(T)) + S(y + N(T)).$$

Now we want to prove that

$$S^{-1} : M \to X/N(T); y \mapsto x + N(T)$$

where $y = T(x)$ for some $x \in X$, is well defined. Such $x \in X$ exists by Definition of M. Now if $T(x_1) = T(x_2)$, then since $T \in L(X,Y)$, $0 = T(x_1 - x_2) = T(x_1) - T(x_2)$ implying that $x_1 + N(T) = x_2 + N(T)$. There is almost nothing to prove that $S \circ S^{-1} = Id : M \to M$ and $S^{-1} \circ S = Id : X/N(T) \to X/N(T)$. The proof of the linearity of S^{-1} is left for readers as an exercise. Since M is a closed subspace of Y and Y is a Banach space, by the fact that closed subspaces of a Banach space is a Banach space, we know that M is a Banach space. Since X is a Banach space, by Exercise 5.8 and 5.9, we have $X/N(T)$ is a Banach space. Now by Corollary 5.2.17, to finish the whole proof, it is remains to prove that S^{-1} is bounded. By Proposition 5.2.2, it is enough to prove that S^{-1} is continuous. We choose $U + N(T)$ be open in $X/N(T)$ randomly with U open in X. By Definition of continuity, it remains to prove that $S_*^{-1}(U + N(T))$ open in M. Since X and Y be Banach space and $T \in L(X, M)$ surjective, by open mapping Theorem, we know T is open. Since U open in X, by Definition 5.2.14, we know that $T(U)$ open in M. So now it remains to prove that

$$(A :=) S_*^{-1}(U + N(T)) = T(U)$$

For $y \in A$, by Definition of preimages and S^{-1}, we know that $x \in U$ for some $x \in X$ where $T(x) = y$, then $y \in T(U)$ by Definition of images. For $y \in T(U)$, by Definition of images, we know $y = T(x)$ for some $x \in U$, then $S^{-1}(y) = x + N(T)$ by Definition of S^{-1}, which implies that $y \in S_*^{-1}(U + N(T))$ by Definition of preimages. $\square$

Question 5.18: Let X be a Banach space and $S \subset X$ such that $\sup_{x \in S} |f(x)| < \infty$ for all $f \in X^*$. Prove that $\sup_{x \in S} \|x\| < \infty$.

Proof. We choose $x \in S$ randomly. Now we define $\overline{x} : X^* \to \mathbb{R}$ by $\overline{x}(f) = f(x)$. Now we want to apply Theorem 5.2.23 called Uniform Boundedness Principle to it. We denote $\mathcal{A} := \{\overline{x} : X^* \to \mathbb{R} : x \in S\}$. Now we want to check that $\overline{x} \in L(X^*, \mathbb{R})$ for each $x \in S$. The checking that $\overline{x}$ is linear is left for readers. We only check that $\overline{x}$ is bounded, which is

$$\sup_{f \in X^*} |\overline{x}(f)| < \infty$$

For each $f \in X^*$, we have $|\overline{x}(f)| = |f(x)| \leq \sup_{a \in S} |f(a)| < \infty$ since $x \in S$. This implies that $\sup_{f \in X^*} |\overline{x}(f)| \leq \sup_{a \in S} |f(a)|$. So we proved that $\mathcal{A} \subset L(X^*, \mathbb{R})$. Furthermore for each $f \in X^*$, we have

$$\sup_{\overline{x} \in \mathcal{A}} |\overline{x}(f)| \leq \sup_{f \in X^*} |\overline{x}(f)| < \infty$$

Since X is a Banach space and $\mathbb{R}$ is a normed space, by Theorem 5.2.23, we have since $\{\overline{x} : x \in S\} = \{\overline{x} : \overline{x} \in \mathcal{A}\}$ by our Definition of $\mathcal{A}$

$$\sup_{x \in S} \|\overline{x}\| = \sup_{\overline{x} \in \mathcal{A}} \|\overline{x}\| < \infty$$

Now to finish the proof, it remains to show that for each $x \in S$. $\|\overline{x}\| = \|x\|$. By Theorem 5.2.10(d), Lemma 0.1 and Definition 5.2.6, we know that $\|\overline{x}\| = \|T(x)\| \leq \|T\|\|x\| = 1 \cdot \|x\|$ where we denote $T : x \mapsto \overline{x}$. Also by Definition of operator norms, we have $\|\overline{x}\| = \sup_{f \in X^* \text{ and } \|f\|=1} |\overline{x}(f)| \geq |\overline{x}(g)| = \|x\|$ where we denote $g : x \mapsto \|x\|$ and such $g \in X^*$ exists given by Theorem 5.2.10(b).

$\square$

Homework 8 — Math 5323

Due: Fridays, 25 March 2022, by 11:59 p.m. CDT

Question 5.19: Let X and Y be Banach spaces and $T : X \to Y$ be a linear operator . Prove that if $f \circ T \in X^*$ for every $f \in Y^*$, then T is bounded.

Proof. Since X, Y are both a Banach space and T is a linear operator, by Theorem 5.2.22, it remains to prove that T is closed. By Definition 5.2.21, we are required to prove that

$$\Gamma(T) = \{(x, y) \in X \times Y : y = T(x)\}$$

is closed in $X \times Y$. We choose a sequence $(x_k, y_k) \in \Gamma(T)$ with $(x_k, y_k) \to (x, y)$ in $X \times Y$. By Definition of closeness, we are required to prove that $(x, y) \in \Gamma(T)$, i.e. $y = T(x)$. By the product topology, we have $x_k \to x$ in X and $y_k \to y$ in Y. We choose $f \in Y^*$ randomly. By the given condition, we have $f \circ T \in X^*$ which says that $f \circ T$ is continuous. So we have

$$f(T(x_k)) = f \circ T(x_k) \to f \circ T(x) = f(T(x)) \tag{1}$$

By the continuity of f, we have by Definition of $\Gamma(T)$

$$f(T(x_k)) = f(y_k) \to f(y) \tag{2}$$

Since $\mathbb{R}$ is Hausdorff, by the uniqueness of limits, (1) and (2) implies that $f(y) = f(T(x))$. But Such $f \in^*$ was chosen randomly, by Lemma 0.1 we must have $y = f(x)$. So far we finished the proof. □

Lemma 0.1. *Let X be a real vector space and $x, y \in X$. Prove that if $f(x) = f(y)$ for any $f \in Y^*$, then $x = y$.*

Proof. We argue this by contradiction and suppose that $x \neq y$. Then by Theorem 5.2.10(c), there is $g \in X^*$ such that $g(x) \neq g(y)$, which gives us a contradiction. □

Question 5.20: Let X and Y be Banach spaces and $\{T_n\} \subset L(X,Y)$ such that $\lim_{n\to\infty} T_n(x)$ exists for every $x \in X$. Define $T(x) = \lim_{n\to\infty} T_n(x)$. Prove that T is a bounded linear operator from X to Y.

Proof. We denote $\mathcal{A} = \{T_n : n = 1, \cdots, \}$. Then we have $\mathcal{A} \subset L(X,Y)$ by the given condition. Now for each $x \in X$, since converge sequences must be bounded, $\lim_{n\to\infty} T_n(x)$ exists implies that $\sup_{n\in\mathbb{N}} ||T_n(x)\| < \infty$. So we have for each $x \in X$, $\sup_{T_n\in\mathcal{A}} \|T_n(x)\| < \infty$. Then, since X be a Banach space and Y be a normed space, by Theorem 5.2.23, we have $\sup_{T_n\in\mathcal{A}} \|T_n\| < \infty$. We choose $x \in X$ with $\|x\| = 1$ randomly. For each $n \in \mathbb{N}$, by Definition of sup, we have

$$\|T_n(x)\| \leq \sup_{T_l\in\mathcal{A}} \|T_l(x)\| (=: M),$$

for some $M \in \mathbb{R}$, which immediately implies that after pushing both sides into ∞, we have by Definition of T and the continuity of norms

$$\|T(x)\| = \| \lim_{n\to\infty} T_n(x)\| = \lim_{n\to\infty} \|T_n(x)\| \leq M$$

But such $x \in X$ was chosen randomly. So we proved that T is bounded by Exercise 5.3.

$\square$

Question 5.21: Let H be an inner product space, Prove that $S^{\perp} = \overline{S}^{\perp}$ for any $S \subset H$.

Proof. We choose $S \subset H$ randomly. There is no doubt that $\overline{S}^{\perp} \subset S^{\perp}$ by Definition 5.3.7 since $S \subset \overline{S}$. Now it remains to prove that $S^{\perp} \subset \overline{S}^{\perp}$. But we have by Remark 5.4 and $S \subset \overline{S}$

$$S^{\perp} = \bigcap_{y \in S} f_y^{-1}(\{0\}) \subset \bigcap_{y \in \overline{S}} f_y^{-1}(\{0\}) = \overline{S}^{\perp}$$

So we finished the proof. $\square$

Question 5.22: Let M be a subspace of a Hilbert space H. Prove that

$$M^{\perp\perp} = \overline{M}$$

Proof. Since H is a Hilbert space, by Definition 5.3.4, we have H is an inner product space. By Question 5.21, we have $M^{\perp} = \overline{M}^{\perp}$ and hence $M^{\perp\perp} = \overline{M}^{\perp\perp}$. Now it remains to prove that $\overline{M} = \overline{M}^{\perp\perp}$. Since M is a subspace, then $\overline{M}$ is a closed subspace. So by Lemma 0.2,, we finish the proof immediately. □

Lemma 0.2. *Let M be a closed subspace of a Hilbert space H. Prove that*

$$M = M^{\perp\perp}$$

Proof. We choose $a \in M$ randomly. We choose $b \in M^{\perp}$ randomly. By Definition 5.3.7, we have

$$\langle a, b\rangle = 0.$$

But such $b \in M^{\perp}$ was chosen randomly. So by Definition 5.3,7 we proved that $a \in M^{\perp\perp}$. But such $a \in M$ was chosen randomly. So we proved that $M \subset M^{\perp\perp}$. We choose $a \in M^{\perp\perp}$ randomly. Since M is a closed subspace of a Hilbert space H , by Theorem 5.3.9, we can write $a = b_1 + b_2$ for some $b_1 \in M$ and $b_2 \in M^{\perp}$. Then by Definition 5.3.7 and Definition 5.3.1.

$$\langle b_2, b_2\rangle = \langle a - b_1, b_2\rangle = \langle a, b_2\rangle - \langle b_1, b_2\rangle = 0 - 0 = 0$$

which implies that $b_2 = 0$ by Definition 5.3.1(iii). So we have $a = b_1 \in M$. But such $a \in M^{\perp\perp}$ was chosen randomly. So we proved that $M^{\perp\perp} \subset M$ and hence we finished proof. □

Question 5.23: Let H be a Hilbert space and $T \in L(H,H)$. Prove the following statements.

a. There is a unique $T^* \in L(H,H)$, called the adjoint of T, such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in H$, $T^{**} = T$, and $\|T^*\| = \|T\|$.(Hint: Theorem 5.3.10)

b. $range(T)^\perp = null(T^*)$ and $null(T)^\perp = \overline{range(T^*)}$.

Proof. a. We are only interested in the real normed space. We first prove the existence. We choose $y \in H$ randomly. We define the map $l_y : H \to \mathbb{R}$ by $x \mapsto \langle T(x), y\rangle$. We check its linearity. For $\alpha \in \mathbb{R}$, $x_1, x_2 \in H$, by the linearity of T and Definition of inner product spaces, $l_y(\alpha x_1 + x_2) = \langle T(\alpha z_1 + z_2), y\rangle = \alpha\langle T(z_1), y\rangle + \langle T(z_2), y\rangle = \alpha l_y(z_1) + l_y(z_2)$. We check its boundedness. For any $x \in H$ with $\|x\| = 1$, we have by Schwartz inequalities and Definition of operator norms, $|l_y(x)| = |\langle T(x), y\rangle| \leq \|T(x)\|\|y\| \leq \|T\|\|x\|\|y\| = \|T\|\|y\|$. Now by the Riesz-Frecher theorem, there is a unique $z \in H$ with $l_y(x) = \langle x, z\rangle$ for all $x \in H$, with $\|l_y\| = \|z\|$. We define $T^*(y) = z$ by the above relations. The uniqueness of z ensures the this map is well-defined a linear operator. We only check that $\|T^*\| < \infty$. For $y \in H$ with $\|y\| = 1$, we have

$$\boxed{\|z\| = \|l_y\| = \sup_{x\in H \text{ and } \|x\|=1} |\langle T(x), y\rangle| \leq \sup_{x\in H \text{ and } \|x\|=1} \|T(x)\|\|y\| \leq \sup_{x\in H \text{ and } \|x\|=1} \|T\|\|x\| = \|T\| < \infty}$$

where the first is due to Riesz-Frechet theorem, the second is due to Definition of operator norms, the third is by Schwartz inequalities, the forth is by Definition of operator norms, and the last is due to $T \in L(H,H)$. So we proved that $T^* \in L(H,H)$. Now we want to prove the uniqueness. We choose $T_1 \in L(H,H)$ satisfying this condition. We choose $y \in H$ randomly. Then

$$\langle T(x), y\rangle = \langle x, T_1(y)\rangle \text{for all } x \in H \tag{3}$$

Also we have

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle \text{ for all } x \in H \tag{4}$$

Combing (3) and (4), we have

$$0 = \langle x, T_1(y) - T^*(y)\rangle \text{ for all } x \in H,$$

which gives that

$$0 = \langle T_1(y) - T^*(y), T_1(y) - T^*(y)\rangle$$

which immediately implies that $T_1(y) = T^*(y)$ by Definition 5.3.1(iii). But such $y \in H$ was chosen randomly. We finished the proof of the uniqueness of the adjoint. We choose $y \in H$ randomly. Then by Definition 0.2.1 and Definition 5.3.1(ii), we have

$$\langle T^*(x), y\rangle = \langle x, T^{**}(y)\rangle \text{ for any } x \in H \tag{5}$$

and

$$\langle x, T(y)\rangle = \langle T^*(x), y\rangle \text{ for any } x \in H. \tag{6}$$

Combing (5) and (6), we have

$$\langle x, T(y) - T^{**}(y)\rangle \text{ for any } x \in H,$$

which implies that after plugging $x = T(y) - T^{**}(y)$, we have

$$0 = \langle T(y) - T^{**}(y), T(y) - T^{**}(y)\rangle$$

which implies that by Definition 5.3.1(iii), $T(y) = T^{**}(y)$. But such $y \in H$ was chosen. We proved that $T = T^{**}$. By the inequality in the box and definition of operator norms, we proved that $\|T^*\| \leq \|T\|$. Then similarly, we have $\|T^{**}\| \leq \|T^*\|$. Then furthermore by $T = T^{**}$, we have $\|T^*\| \leq \|T\| = \|T^{**}\| \leq \|T^*\|$, which implies immediately that $\|T\| = \|T^*\|$.

b. First we prove that $range(T)^\perp = null(T^*)$. For any $x \in range(T)^\perp$,

$$\langle T^*(x), T^*(x)\rangle = \langle T(T^*(x)), x\rangle = 0$$

where the first inequality is due to Definition 0.2.1 and the second is due to Definition 5.3.6 and Definition of images of maps. This implies that we proved that $range(T)^{\perp} \subset null(T^*)$. For any $x \in null(T^*)$

$$\langle z, x\rangle = \langle T(y), x\rangle = \langle y, T^*(x)\rangle = \langle y, 0\rangle = 0$$

holds for any $z \in range(T)$ we denote $z = T(y)$ for some $y \in H$, where the first equality is due to Definition of range of maps, the second is due to Definition 0.2.1, the third is due to Definition of null of maps and the last is due to the sequence of Definition 5.3.1(i). This implies that $null(T^*) \subset range(T^{\perp})$ by Definition 5.3.7.

We prove that $null(T)^{\perp} = \overline{range(T^*)}$. Actually we have

$$null(T)^{\perp} = null(T^{**})^{\perp} = range(T^*)^{\perp\perp} = \overline{range(T^*)}$$

where the first is due to (a), the second is due to the previous result and the third is due to Question 5.22 since $range(T^*)$ is a closed subspace of the Hilbert space H by the knowledge of basic linear algebra.

□

Definition 0.2.1. *Let H be a Hilbert space and $T \in L(H, H)$. There is a unique $T^* \in L(H, H)$, called the adjoint of T, such that*

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

holds for any $x, y \in H$.

Homework 9 — Math 5323
Due: Fridays, 1 April 2022, by 11:59 p.m. CDT

Question 5.24: Let K be a closed convex set in a Hilbert space H. Prove that

a. Any sequence in K whose norms approach to $\inf_{x\in K}\|x\|$ is a Cauchy sequence, and

b. K has a unique element of minimal norm.

(Hint: The parallelogram law.)

Proof. a. We randomly choose a sequence $x_n \in K$ such that $\|x_n\| \to \inf_{x\in K}\|x\|$ in $\mathbb{R}$. We choose $\epsilon > 0$ randomly. By Definition of limits, we have $\inf_{x\in K}\|x\| \in \mathbb{R}$. We denote $d := \inf_{x\in K}\|x\|$ for some $d \in \mathbb{R}$. But actually we have

$$\|x_m - x_n\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - 4\left\|\frac{x_n + x_m}{2}\right\|^2 \leq 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2 \tag{1}$$

where the first is due to Theorem 5.3.6. and the second is due to Definition of convex sets which gives that $x_n, x_m \in K$ implies that $\frac{x_n+x_m}{2} \in K$ and Definition of inf. By Definition of limits, this gives a $N_1 \in \mathbb{N}$ such that

$$\|x_n\|^2 < \frac{\epsilon + 4d^2}{4} \tag{2}$$

holds for any $n \in \mathbb{N}$ with $n \geq N_1$. Similarly, we have a $N_2 \in \mathbb{N}$ such that

$$\|x_m\|^2 < \frac{\epsilon + 4d^2}{4} \tag{3}$$

holds for any $m \in \mathbb{N}$ with $m \geq N_2$. Then Taking $N := \max\{N_1, N_2\}$, combing (1), (2) and (3), we have

$$\|x_m - x_n\|^2 < 2\tfrac{\epsilon+4d^2}{4} + 2\tfrac{\epsilon+4d^2}{4} - 4d^2 = \epsilon$$

holds for any $m, n \in \mathbb{N}$ with $m, n \geq N$. But such $\epsilon > 0$ was chosen randomly. By Definition of Cauchy sequences, we finished the proof.

b. First we prove the uniqueness. We choose $x, y \in H$ such that $\|y\| = \|x\| = \inf_{z\in K}\|z\|$. We denote $d := \inf_{z\in K}\|z\|$. Then (1) gives that

$$\|x - y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - 4d^2 = 2d^2 - 2d^2 - 4d^2 = 0,$$

which immediately implies that $x = y$ by Definition of norms. We prove the existence.

$\square$

Question 5.25: Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_n\}$ be a partition of X. Let $f_{nk} \in L^2(\mu)$ with $n, k \in \mathbb{N}$ be a collection of measurable functions. Prove that if $\{f_{nk}\chi_{E_n} : k \in \mathbb{N}\}$ is an orthonormal basis for $L^2(E_n, \mu)$, $n = 1, 2, \cdots$, then $\{f_{nk} : n, k \in \mathbb{N}\}$ is an orthonormal basis for $L^2(\mu)$.

Proof. We denote $\mathcal{B} := \{f_{nk} : n, k \in \mathbb{N}\}$. We prove that $\mathcal{B}$ collects unit vectors. We choose $f_{nk} \in \mathcal{B}$ randomly. Then

$$\begin{aligned}
\|f_{nk}\|^2 &= \langle f_{nk}, f_{nk}\rangle && (4)\\
&= \int_X f_{nk} f_{nk} d\mu && (5)\\
&= \int_{\bigcup_{p=1}^{\infty} E_p} f_{nk} f_{nk} d\mu && (6)\\
&= \sum_{p=1}^{\infty} \int_{E_p} f_{nk} f_{nk} d\mu && (7)\\
&= \int_{E_n} f_{nk}\chi_n f_{nk}\chi_{nk} d\mu && (8)\\
&= \langle f_{nk}\chi_n, f_{nk}\chi_n\rangle_{E_n} && (9)\\
&= 1 && (10)
\end{aligned}$$

where (4) is by Definition 5.3.1, (5) is by Example 5.4(e), (6) is by Definition of a partition, (7) is by properties of integration, (8) is by the given condition, (9) is by Example 5.4(e) and (10) is by Definition 5.3.11 and Definition 5.3.14, which implies that $\|f_{nk}\| = 1$ immediately.
We prove that elements of $\mathcal{B}$ are orthogonal with each other. We choose $f_{nk}, f_{pl} \in \mathcal{B}$ such that $(n, k) \neq (p, l)$ randomly. Then

$$\begin{aligned}
I := \langle f_{nk}, f_{pl}\rangle &= \int_X f_{nk} f_{pl} d\mu && (11)\\
&= \sum_{m=1}^{\infty} \int_{E_m} f_{nk} f_{pl} d\mu && (12)\\
&= \sum_{m=1}^{\infty} \int_{E_m} f_{nk}\chi_{E_m} f_{pl}\chi_{E_m} d\mu && (13)
\end{aligned}$$

Now if $n \neq p$, (13) says $I = 0$. We consider $n = p$ then $k \neq l$ since $(n, k) \neq (p, l)$. Then (13) says

$$I = \int_{E_n} f_{nk}\chi_{E_n} f_{nl}\chi_{E_n} d\mu = \langle f_{nk}\chi_{E_n}, f_{nl}\chi_{E_n}\rangle_{E_n} = 0$$

where the last equality is due to Definition 5.3.11 and Definition 5.3.14 with $k \neq l$. Finally we prove the completeness and hence by Definition 5.3.14, Definition 5.3.11 and Theorem 5.3.13, we finish the proof. We choose $g \in L^2(\mu)$ such that $\langle g, f_{nk}\rangle = 0$ for any $n, k \in \mathbb{N}$ randomly. We fix $n \in \mathbb{N}$ randomly. Then we have

$$0 = \langle g, f_{nk}\rangle = \langle g\chi_{E_n}, f_{nk}\chi_{E_n}\rangle_{E_n}$$

holds for any k, which says that

$$g\chi_{E_n} = 0$$

by Theorem 5.3.13 and Definition 5.3.14. But such $n \in \mathbb{N}$ was chosen randomly. So we have $g\chi_{E_n} = 0$ for each $n \in \mathbb{N}$. Then by Definition of characteristic functions and Definition of partitions

$$g = g\chi_X = g\chi_{\bigcup_{n=1}^{\infty} E_n} = g(\sum_{n=1}^{\infty} \chi_{E_n}) = \sum_{n=1}^{\infty} g\chi_{E_n} = \infty \cdot 0 = 0$$

By Theorem 5.3.13(a), we proved the completeness. □

Question 5.26: Let $\{f_m\}$ and $\{g_n\}$ be orthonormal bases for $L^2(\mu)$ and $L^2(\nu)$ over $\sigma-$finite measure space $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$, respectively. Prove that $\{h_{mn} = f_m(x)g_n(y)\}$ is an orthonormal basis for $L^2(\mu \times \nu)$

Proof. We denote $\mathcal{B} := \{h_{(m,n)} : m, n \in \mathbb{N}\}$, where we define $h_{(m,n)} : X \times Y \to \mathbb{R}; (x, y) \mapsto f_m(x)g_n(y)$ for each $m, n \in \mathbb{N}$. We prove that $\mathcal{B} \subset L^2(\mu \times \nu)$ is orthonormal. We choose $h_{mn}, h_{kl} \in \mathcal{B}$ with $(m, n) \neq (k, l)$. Then we have

$$\begin{aligned}
\langle h_{mn}, h_{kl}\rangle &= \int_{X\times Y} h_{mn}(x, y)h_{kl}(x, y)d(\mu \times \nu) && (14)\\
&= \int_{X\times Y} f_m(x)g_n(y)f_k(x)g_l(y)d(\mu \times \nu) && (15)\\
&= \int_{X\times Y} \Big(f_m(x)f_k(x)\Big)\Big(g_n(y)g_l(y)\Big)d(\mu \times \nu) && (16)\\
&= \Big(\int_X f_m(x)f_k(x)d\mu\Big)\Big(\int_Y g_n(y)g_l(y)d\nu\Big) && (17)\\
&= \langle f_m, f_n\rangle\langle g_n, g_k\rangle && (18)\\
&= 0 && (19)
\end{aligned}$$

where (4) is due to Definition of $L^2(\mu \times \nu)$, (5) is due to Definition of h_{mn}, (7) is by Fubini Theorem, (8) is due to Definition of $L^2(\mu)$ and $L^2(\nu)$ and (9) is by our choice of h_{mn} and h_{kl} and Definition 5.3.11 and Definition 5.3.14 where $\{f_m\}$ and $\{g_n\}$ are orthonormal bases for $L^2(\mu)$ and $L^2(\nu)$ respectively. We choose $h_{mn} \in \mathcal{B}$ randomly. Then we have

$$\begin{aligned}
\langle h_{mn}, h_{mn}\rangle &= \int_{X\times Y} h_{mn}(x, y)h_{mn}(x, y)d(\mu \times \nu) && (20)\\
&= \int_{X\times Y} f_m(x)g_n(y)f_m(x)g_n(y)d(\mu \times \nu) && (21)\\
&= \int_{X\times Y} f_m^2(x)g_n^2(y)d(\mu \times \nu) && (22)\\
&= \Big(\int_X f_m^2(x)d\mu\Big)\Big(\int_Y g_n^2(y)d\nu\Big) && (23)\\
&= \langle f_m, f_m\rangle\langle g_n, g_n\rangle && (24)\\
&= 1 \cdot 1 && (25)\\
&= 1 && (26)
\end{aligned}$$

where (10) is due to Definition of $L^2(\mu \times \nu)$, (11) is due to Definition of h_{mn}, (13) is by Fubini Theorem, (14) is by Definition of $L^2(\mu)$ and $L^2(\nu)$, (15) is by Definition 5.3.11 and Definition 5.3.14 where $\{f_m\}$ and $\{g_n\}$ are orthonormal bases for $L^2(\mu)$ and $L^2(\nu)$ respectively. We prove that $\mathcal{B}$ is a basis for $L^2(\mu \times \nu)$. We randomly choose $\phi \in L^2(\mu \times \nu)$ such that $\langle \phi, h_{mn}\rangle = 0$ for any $m, n \in \mathbb{N}$. Then for any $m, n \in \mathbb{N}$, we have

$$\begin{aligned}
0 = \langle \phi, h_{mn}\rangle &= \int_{X\times Y} \phi(x, y)h_{mn}(x, y)d(\mu \times \nu) && (27)\\
&= \int_Y \int_X \phi(x, y)h_{mn}(x, y)d\mu(x)d\nu(y) && (28)\\
&= \int_Y \int_X \phi^y(x)f_m(x)g_n(y)d\mu(x)d\nu(y) && (29)\\
&= \int_Y g_n(y)\Big(\int_X \phi^y(x)f_m(x)d\mu(x)\Big)d\nu(y) && (30)\\
&= \int_Y g_n(y)\langle \phi^y, f_m\rangle d\nu(y) && (31)\\
&= \int_Y g_n(y) \cdot 0d\nu(y) && (32)\\
&= 0 && (33)
\end{aligned}$$

where we denote $\phi^y(x) := \phi(x, y)$ for each $y \in Y$ where (17) is due to Definition of $L^2(\mu \times \nu)$, (18) is by Fubini's Theorem, (21) is by Definition of $L^2(\mu)$ and (22) is By Fubini Theorem which implies that $\phi^y \in L^2(\mu)$ for each $y \in Y$ and Definition 5.3.14. Finally by Definition 5.3.14, we proved that $\mathcal{B}$ is an orthonormal basis for $L^2(\mu \times \nu)$. □

Question 5.27: Prove the following statements

a. The set of all polynomials is dense in $L^2([0,1],m)$.(Hint: Theorems 6.1.8 and 4.5.6)

b. $L^2([0,1],m)$ is separable. (Hint: Proposition 5.3.16)

c. $L^2(\mathbb{R},m)$ is separable.(See Exercise 5.25)

d. $L^2(\mathbb{R}^n,m)$ is separable. (See Exercise 5.26)

Proof. a. We denote $\mathcal{P} := \left\{p : [0,1] \to \mathbb{R} \middle| p(x) = a_0 + a_1 x + \cdots + a_n x^n \text{ where } a_n, \cdots, a_0 \in \mathbb{R} \text{ with } a_n \neq 0 \text{ and } n \in \mathbb{N}\right\}$.
We denote $C_c([0,1],\mathbb{R}) = \left\{f : [0,1] \to \mathbb{R} \text{ continuous} \middle| \exists \text{ a compact } K \subset [0,1] \text{ such that } f(x) = 0 \ \forall x \in [0,1] \setminus K\right\}$.
By Theorem 6.1.8 with $n = 1$ and $p = 2$, we have

$$\overline{C_c([0,1],\mathbb{R})} = L^2([0,1],m) \tag{34}$$

We want to use Theorem 4.5.6 to prove that

$$\overline{\mathcal{P}} = C([0,1],\mathbb{R}) = C_c([0,1],\mathbb{R}) \tag{35}$$

There is no doubt that $\overline{\mathcal{P}} \subset C([0,1],\mathbb{R})$ since polynomials are always continuous. There is no doubt that $[0,1]$ is a compact Hausdorff space since a subspace of a Hausdorff space is always Hausdorff and any closed and bounded subset of $\mathbb{R}$ is compact due to Heine–Borel theorem. We prove that $\overline{\mathcal{P}}$ is a subalgebra. The checking that $\mathcal{P}$ is a subspace of $C([0,1],\mathbb{R})$ over $\mathbb{R}$ is left for readers. There is no doubt that the multiplication of polynomials is still a polynomial. There is no doubt that $\overline{\mathcal{P}}$ separates points in $[0,1]$ and quadratic functions can finish this work. Also the constant function is in $\overline{\mathcal{P}}$ since $([0,1] \ni z \mapsto p(z) := 1 \in \mathbb{R})$ is a polynomial. Now by Theorem 4.5,6 we prove (25). Combing (24) and (25), by Definition of dense and Definition 4.1.1, we proved that $\mathcal{P}$ is dense in $L^2([0,1],m)$.

b. Since $L^2([0,1],m)$ is a Hilbert space, by Proposition 5.3.16, it is equivalent to prove that $L^2([0,1],m)$ has a countable orthonormal basis. We finish this question by a constructive proof. We denote $\mathcal{B}_1 = \{1, x, x^2, \cdots\}$. We apply the Gram-Schmidt orthogonalization to $\mathcal{B}_1$ to get its orthonormal set. We do it in the following inductive way. We set

$$\boldsymbol{u_1} := 1 \qquad \boldsymbol{e_1} := \frac{\boldsymbol{u_1}}{\|\boldsymbol{u_1}\|} = \frac{1}{1} = 1$$

$$\boldsymbol{u_2} := \boldsymbol{v_2} - proj_{\boldsymbol{u_1}}(\boldsymbol{v_2}) = \boldsymbol{v_2} - \frac{\langle \boldsymbol{u_1}, \boldsymbol{v_2}\rangle}{\langle \boldsymbol{u_1}, \boldsymbol{u_1}\rangle}\boldsymbol{u_1} = x - \int_{[0,1]} x d\mu(x) = x - \frac{1}{2} \qquad \boldsymbol{e_2} := \frac{1}{12}(x - \tfrac{1}{2})$$

$$\boldsymbol{u_3} := \boldsymbol{v_3} - proj_{\boldsymbol{u_1}}(\boldsymbol{v_3}) - proj_{\boldsymbol{u_2}}(\boldsymbol{v_3}) = \boldsymbol{v_3} - \tfrac{\langle \boldsymbol{u_1}, \boldsymbol{v_3}\rangle}{\langle \boldsymbol{u_1}, \boldsymbol{u_1}\rangle}\boldsymbol{u_1} - \tfrac{\langle \boldsymbol{u_2}, \boldsymbol{v_3}\rangle}{\langle \boldsymbol{u_2}, \boldsymbol{u_2}\rangle}\boldsymbol{u_2} = x^2 - x + \tfrac{1}{6} \qquad \boldsymbol{e_3} := x^4 - 2x^3 + \tfrac{4}{3}x^2 - \tfrac{1}{3}x + \tfrac{1}{36}$$

By this process, there is no doubt that $\langle \boldsymbol{e_i}, \boldsymbol{e_j}\rangle = 1$ for $i \neq j$ and $\langle \boldsymbol{e_i}, \boldsymbol{e_i}\rangle = 1$ for all i. We have that

$$\boldsymbol{u}_n = \boldsymbol{v}_n - \sum_{l=1}^{n-1} \frac{\langle \boldsymbol{u}_l, \boldsymbol{v}_n\rangle}{\langle \boldsymbol{u}_l, \boldsymbol{u}_l\rangle}\boldsymbol{u}_l \tag{36}$$

and we have

$$\boldsymbol{e}_n = \frac{\boldsymbol{u}_n}{\sqrt{\langle \boldsymbol{u}_n, \boldsymbol{u}_n\rangle}} \tag{37}$$

We denote $\mathcal{B}_2 := \{\boldsymbol{e_1}, \cdots, \}$. We choose $g \in C([0,1],\mathbb{R})$ randomly. By Definition 5.3.14 and Theorem 5.3.13(b), it remains to prove that

$$\|g\|^2 = \sum_{n=1}^{\infty} \left|\langle g, \boldsymbol{e}_m\rangle\right|^2 \tag{38}$$

By (a), we can choose a sequence $p_l \in \mathcal{P}$ such that

$$\lim_{l\to\infty} p_l = g$$

So now it is enough to prove that for each $l \in \mathbb{N}$,

$$\|p_l\|^2 = \sum_{n=1}^{\infty} \left|\langle p_l, \boldsymbol{e}_n\rangle\right|^2 \tag{39}$$

since by the continuity of norms and the serious converges where we can exchange limits freely

$$\|g\|^2 = \|\lim_{l\to\infty} p_l\|^2 = \lim_{l\to\infty}\|p_l\|^2 = \lim_{l\to\infty}\sum_{n=1}^{\infty}\left|\langle p_l, \boldsymbol{e}_n\rangle\right|^2 = \sum_{n=1}^{\infty}\left|\langle \lim_{l\to\infty} p_l, \boldsymbol{e}_n\rangle\right|^2 = \sum_{n=1}^{\infty}\left|\langle p, \boldsymbol{e}_n\rangle\right|^2$$

For making life easier, we denote $p := p_l$. Now it is enough to prove the assertion:

$$(A :=)\|p\|^2 = \sum_{n=1}^{\infty}\left|\langle p, \boldsymbol{e}_n\rangle\right|^2 (=: B)$$

where $\boldsymbol{e}_n$ is given by (26), (27) and $\boldsymbol{v}_n = x^n$ for each $n \in \mathbb{N}$. we denote

$$p(x) = a_0 + a_1x^1 + a_2x^2 + \cdots + a_nx^n$$

for some $n \in \mathbb{N}$ with $n \neq 0$, and $a_0, \cdots, a_n \in \mathbb{R}$ with $a_n \neq 0$. Now we want to prove this by induction on $degree(p)$. For the base step which $degree(p) = 0$, we have by Definition 5.3.1

$$A = \langle a_0, a_0\rangle = a_0^2\langle 1, 1\rangle = a_0^2$$

and we also have

$$B = \sum_{n=1}^{\infty}\left|\langle a_0, \boldsymbol{e}_n\rangle\right|^2 = \sum_{n=1}^{\infty}\left|a_0\langle 1, \boldsymbol{e}_n\rangle\right|^2 = a_0^2\sum_{n=1}^{\infty}\left|\langle 1, \boldsymbol{e}_n\rangle\right|^2 = a_0^2\sum_{n=1}^{\infty}\left|\langle 1, \frac{\boldsymbol{u}_n}{\sqrt{\langle \boldsymbol{u}_n, \boldsymbol{u}_n\rangle}}\rangle\right|^2$$

This is left as an exercise left for readers. From the previous discussion, we know that it is now enough to prove that $\mathcal{B}$ spans $\mathcal{P}$ By Gram-Schmidt process and the knowledge of linear algebra, we know that if $\mathcal{B}_1$ spans $\mathcal{P}$, then $\mathcal{B}$ spans $\mathcal{P}$. Now it is enough to prove that $\mathcal{B}_1$ spans $\mathcal{P}$ by Definition 0.0.1. We choose $p \in \mathcal{P}$ such that $\langle p, x^l\rangle = 0$ for each $l \in \mathbb{N}$. By Theorem 5.3.13 and Definition 5.3.14, it remains to prove that $p = 0$. We denote $p(x) = a_0 + a_1x + \cdots + a_nx^n$ for some $a_0, \cdots, a_n \in \mathbb{R}$ with $a_n \neq 0$ where $n := degree(p)$. Now by our choice of p, we have

$$\langle p, 1\rangle = 0, \cdots, \langle p, x^n\rangle = 0$$

Then by Definition 5.3.1, we have

$$0 = a_0\langle p, 1\rangle = \langle p, a_0\rangle, \cdots, a_n\langle p, x^n\rangle = \langle p, a_nx^n\rangle = 0$$

Adding them together, by Definition 5.3.1, we have

$$0 = \langle p, a_0\rangle + \cdots + \langle p, a_nx^n\rangle = \langle p, a_0 + \cdots + a_nx^n\rangle = \langle p, p\rangle$$

which immediately implies that by Definition 5.3.1. $p = 0$.

c. Since $L^2(\mathbb{R}, m)$ is a Hilbert space, by Proposition 5.3.16, it is equivalent to prove that $L^2(\mathbb{R}, m)$ has a countable orthonormal basis. We fix $n \in \mathbb{Z}$ randomly. By (b), we know that $L^2([n, n+1], m)$ is separable. Then by Proposition 5.3.16, $L^2([n, n+1], m)$ has countable orthornormal and we denote it by $\{p_{nl}\big|_{[n,n+1]} : l \in \mathbb{N}\}$. Now by Question 5.25, if we prove that $\{[n, n+1] : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$, then $\{p_{nl} : l \in \mathbb{N}\ n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R}, m)$ and obviously it is countable. There is a little confused here that $\{[n, n+1] : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$ since $[1, 2] \cap [2, 3] \neq \varnothing$. ALso in the previous proof, the compactness of $[n, n+1]$ is critical in the use of Stone-Weierstrass Theorem.

d. Since $A := L^2(\mathbb{R}^n, m)$ is a Hilbert space, by Proposition 5.3.16, it is equivalent to prove that A has a countable orthonormal basis. Now by Question 5.26 and the inductive method, it is enough to prove that $L^2(\mathbb{R}, m)$ has a countable orthonormal basis. Then by Proposition 5.3.16, it is equivalent to prove that $L^2(\mathbb{R}, m)$ is separable, which is given by (c). We denote an orthonormal basis for $L^2(\mathbb{R}^{(i)}, m)$ by

$$\mathcal{B}_i = \left\{f_l^{(i)} : l \in \mathbb{N}\right\}$$

for each $i = 1, \cdots, n$ Now we define from $\mathbb{R}^n$ to $\mathbb{R}$

$$g_{il}(x^{(1)}, \cdots, x^{(n)}) = \prod_{i=1}^{n} f_l^{(i)}(x^{(i)})$$

for each $i = 1, \cdots, n$ and $l \in \mathbb{N}$ Then $\mathcal{B} := \left\{ g_{il} : i = 1, \cdots, n,\ l \in \mathbb{N} \right\}$ is an orthonormal basis for $L^2(\mathbb{R}^n, m)$.

$\square$

Definition 0.0.1. *Let H be an inner product space and Let $\mathcal{B} \subset H$ be a subset. $\mathcal{B}$ is said to span H if it satisfies one of the following equivalent conditions*

a. For any $v \in H$, if $\langle v, a \rangle = 0$ for any $a \in B$, then $v = 0$.

b. For any $v \in H$, $\sum_{a \in \mathcal{B}} |\langle v, a \rangle|^2 = \|v\|^2$

c. For any $v \in H$, $v = \sum_{a \in \mathcal{B}} \langle v, a \rangle a$

Question 5.28: Let H be an infinite-dimensional Hilbert space. Prove the following statements.

a. Every orthonormal sequence in H converges weakly to 0.

b. For any $x \in H$ with $\|x\| < 1$, there is a sequence $\{u_n : n \in \mathbb{N}\}$ of unit vectors such that $u_n \to x$ weakly.

Proof. a. We choose an orthonormal sequence $x_n \in H$ randomly. We choose $f \in H^*$ randomly. Now by Definition 5.2.25 and Definition of normed spaces, it is enough to prove that

$$\left| f(x_n) - f(0) \right| \to 0 \tag{40}$$

Since H is Hilbert space, by Theorem 5.3.10, we can choose $y \in H$ such that

$$f(x) = \langle x, y \rangle \text{ for any } x \in H. \tag{41}$$

Now we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| f(x_n) - f(0) \right|^2 &= \sum_{n=1}^{\infty} \left| \langle x_n, y \rangle - \langle 0, y \rangle \right|^2 && (42)\\
&= \sum_{n=1}^{\infty} \left| \langle x_n, y \rangle \right|^2 && (43)\\
&= \sum_{n=1}^{\infty} \left| \langle y, x_n \rangle \right|^2 && (44)\\
&\le \|y\|^2 && (45)\\
&< \infty && (46)
\end{aligned}$$

where (32) is due to (31), (33) and (34) are due to Definition 5.3.1., (35) is due to Theorem 5.3.12 where $\{x_n : n \in \mathbb{N}\}$ is an orthonormal set, and (36) is due to H is an Hilbert space, which implies that by the knowledge of babe real analysis

$$\lim_{n\to\infty} \left| f(x_n) - f(0) \right|^2 = 0.$$

This proves (30) by the continuity of product functions.

b. We choose $x \in H$ with $\|x\| < 1$ randomly. We denote $E := \{x\}^{\perp}$. There is no doubt that E is a closed subspace of H. Since H is a Hilbert space, by Theorem 0.1 and Definition 5.3.4, we know E is a Hilbert space. Then by Proposition 5.3.15, we know that E has an orthonormal basis. There is no doubt by the knowledge of linear algebra and Definition 5.3.7 that

$$H = span\{x\} \oplus E$$

which implies that $\dim(H) - 1 = \dim(E)$ by Definition of dimension and immediately says that E is of infinite-dimensional since H is of infinite-dimensional. Now we denote $\mathcal{B} := \{z_n \in H : n \in \mathbb{N}\}$ be infinite countable orthonormal basis for E. Now for each $n \in \mathbb{N}$, we denote $u_n = x + a_n z_n$ for some $a_n \in \mathbb{C}$ and we are trying to find $a_n \in \mathbb{R}^+$ such that $\langle u_n, u_n \rangle = 1$. But by Definition 5.3.1 and Definition 5.3.11

$$1 = \langle u_n, u_n \rangle = \|x\|^2 + |a_n|^2$$

which says that we can choose $a_n = \sqrt{1 - \|x\|^2}$ where $\|x\| < 1$. Now we have a sequence $\{u_n = x + \sqrt{1 - \|x\|^2} z_n : n \in \mathbb{N}\}$ of unit vectors. Finally we want to prove that $u_n \to x$ weakly and hence finish the proof. We choose $f \in H^*$ randomly. By Definition 5.2.24, we are require to prove that

$$f(u_n) \to f(x)$$

But we have by linearity of f and (a) where $\{u_n\} \subset H$ is orthonormal

$$\lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} f(x + \sqrt{1 - \|x\|^2} z_n) = f(x) + \sqrt{1 - \|x\|^2} \lim_{n\to\infty} f(z_n) = f(z) + \sqrt{1 - \|x\|^2} \cdot 0 = 0$$

which says that we finished the proof.

□

Theorem 0.1. *A closed subspace of a complete space is still complete.*

Homework 10 — Math 5323
Due: Fridays, 8 April 2022, by 11:59 p.m. CDT

Question 6.2: Let $(X, \mathcal{M}, \mu)$ be a $\sigma-$finite measure space. Let $1 \leq p < q < r \leq \infty$. Prove that $L^p + L^r$ is a Banach space with norm

$$\|f\| = \inf\{\max(\|g\|_p, \|h\|_r) : f = g + h \; g \in L^P \; h \in L^r\}$$

and the inclusion map $L^q \to L^p + L^r$ is continuous. (See Proposition 6.1.13.)

Proof. In this question we are only interested in the real normed vector space and $\sigma-$finite measure space $(X, \mathcal{M}, \mu)$ which implies that $\mu(E) < \infty$ for any bounded set $E \subset X$ with $\mu(X) < 1$. We prove that $A := L^p + L^r$ is a vector space over $\mathbb{R}$. By The proof in the page 175, we know that L^P and L^r are both vector space over $\mathbb{R}$. Since the sum of real vector spaces is still a real vector space, we know that $L^p + L^r$ is a real vector space. We prove that $\| \cdot \|$ is a well defined norm on $L^p + L^r$. Since for any $f \in L^p + L^r$, $\max(\|g\|_p, \|h\|_r) \geq 0$ by Definition of max, we have $\|f\| \geq 0$ by Definition of max. Also by Definition 5.1.1, $\|f\| < \infty$ for any $f \in L^p + L^r$.

(i) We prove for any $f \in A$, $\|f\|$ if and only if $f = 0$. For any $f \in A$ with $f = 0$, we have $\|f\| = \inf\{\max(\|g\|_p, \|h\|_r) : 0 = g + h\} = \inf\limits_{g=h=0} \max(\|g\|_p, \|h\|_r) = 0$ by Definition 5.1.1(i). For any $f \in A$ with $\|f\| = 0$, by Definition of inf, we have sequences $g_n \in L_p$ and $h_n \in L_r$ such that $0 = \lim\limits_{n\to\infty} \max(\|g_n\|_p, \|h_n\|_r)$ which implies that $0 = \lim\limits_{n\to\infty} \|g_n\|_p = \lim\limits_{n\to\infty} \|h_n\|_r$ by Definition of max, and hence by properties of norms and properties of limits $f = \lim\limits_{n\to\infty} f = \lim\limits_{n\to\infty} g_n + \lim\limits_{n\to\infty} h_n = 0 + 0 = 0$.

(ii) We prove that for any $f_1, f_2 \in A$, $\|f_1 + f_2\| \leq \|f_1\| + \|f_2\|$. We choose $f_1, f_2 \in A$ randomly. Then by Definition of inf, we have sequences $g_n^{(1)} \in L^p$ and $h_n^{(1)} \in L^r$ such that $f_1 = g_n^{(1)} + h_n^{(1)}$ for each $n \in \mathbb{N}$ and

$$\|f_1\| = \lim_{n\to\infty} \max(\|g_n^{(1)}\|_p, \|h_n^{(1)}\|_r)$$

Similarly, we have sequences $g_n^{(2)} \in L^p$ and $h_n^{(2)} \in L^r$ such that and $f_2 = g_n^{(2)} + h_n^{(2)}$ for each $n \in \mathbb{N}$ and

$$\|f_2\| = \lim_{n\to\infty} \max(\|g_n^{(2)}\|_p, \|h_n^{(2)}\|_r)$$

Then we have for each $n \in \mathbb{N}$, we have by Definition of max and inequalities of norms

$$\max(\|g_n^{(2)}\|_p, \|h_n^{(2)}\|_r) + \max(\|g_n^{(1)}\|_p, \|h_n^{(1)}\|_r) \geq \max(\|g_n^{(1)} + g_n^{(2)}\|_p, \|h_n^{(1)} + h_n^{(2)}\|_r)$$

which implies that after pushing both sides to ∞, we have

$$\|f_1\| + \|f_2\| \geq \lim_{n\to\infty} \max(\|g_n^{(1)} + g_n^{(2)}\|_p, \|h_n^{(1)} + h_n^{(2)}\|_r) \geq \|f_1 + f_2\|$$

where the last inequality is due to Definition of inf where $f_1 + f_2 = g_n^{(1)} + g_n^{(2)} + h_n^{(1)} + h_n^{(2)}$ for each $n \in \mathbb{N}$ and hence $\max(\|g_n^{(1)} + g_n^{(2)}\|_p, \|h_n^{(1)} + h_n^{(2)}\|_r) \in \left\{ \max(\|g\|_p, \|h\|_r) : f = g + h \right\}$ for each $n \in \mathbb{N}$.

(iii) We prove that for any $\alpha \in \mathbb{R}$, $f \in A$, $\alpha f \in A$. We choose $\alpha \in \mathbb{R}$ and $f \in A$ randomly. We have by Definition of norms in this question, Definition of inf and Definition of max,

$$|\alpha|\|f\| = |\alpha| \inf\{\max(\|g\|_p, \|h\|_r) : f = g + h\} = \inf\{\max(\|\alpha g\|_p, \|\alpha h\|_r) : f = g + h\} \leq \|\alpha f\|.$$

where the last inequality is due to if $f = g + h$ then $\alpha f = \alpha g + \alpha h$. Now we also have

$$\begin{aligned}
\|\alpha f\| &= \inf\{\max(\|g\|_p, \|h\|_r) : \alpha f = g + h\} \\
&= \inf\{\max(\|g\|_p, \|h\|_r) : f = \frac{g}{\alpha} + \frac{h}{\alpha}\} \\
&\leq \inf\{\max(\|\alpha g\|_p, \|\alpha h\|_r) : f = g + h\} \\
&= |\alpha| \inf\{\max(\|g\|_p, \|h\|_h) : f = g + h\} \\
&= |\alpha|\|f\|
\end{aligned}$$

where the first equality is due to Definition in the question, the third inequality is due $\{\max(\|g\|_p, \|h\|_r) : f = \frac{g}{\alpha} + \frac{h}{\alpha}\} \subset \{\max(\|\alpha g\|, \|\alpha h\|_r) : f = g + h\}$ and the last equality is due to Definition in the questions. So far we have $(A, \| \cdot \|)$ is a normed real space.

(iv) We prove the completeness. We choose a Cauchy sequence $f_n \in A$ randomly. We prove that $f_n \to f$ in A for some $f \in A$. By Definition of the set A, we can write

$$f_n = g_n + h_n \tag{1}$$

for some sequences $g_n \in L^P, h_n \in L^r$. Now we have for each $n, m \in \mathbb{N}$ by Definition of norms in the question

$$\|f_n - f_m\| \geq \|g_n - g_m\|_p$$

which says that f_n is Cauchy in A implies that g_n is Cauchy in L^p. By Theorem 6.1.6 and Definition 5.1.2, $g_n \to g$ in L^P for some $g \in L^p$. Similarly we can choose $h \in L^r$ such that $h_n \to h$ in L^r. We know that L^p is embedded in the normed vector space L. So $g_n \to g$ in $(A, \|\cdot\|)$ and hence $\lim_{n\to\infty} \|g_n - g\| = 0$ by the fact that the topology on a normed space is induced by its norm and similarly we have $\lim_{n\to\infty} \|h_n - h\| = 0$. Hence we have

$$\lim_{n\to\infty} \|f_n - (g+h)\| \overbrace{=}^{(i)} \lim_{n\to\infty} \|(g_n + h_n) - (g+h)\| \overbrace{=}^{(ii)} \lim_{n\to\infty} \|g_n - g + h_n - h\| \overbrace{\leq}^{(iii)} \lim_{n\to\infty} \|g_n - g\| + \lim_{n\to\infty} \|h_n - h\| = 0 + 0 = 0$$

where (i) is due to (1) and (iii) is due to triangle inequalities of norms, which immediately implies that

$$f_n \to g + h \text{ in } (A, \|\cdot\|).$$

by the fact that the topology on a normed space is induced by its norm. Also $g \in L^p$ and $h \in L^r$. By Definition of completeness of a topology space, we prove that the normed space $(A, \|\cdot\|)$ is complete and hence by Definition 5.1.2, we proved that $(A, \|\cdot\|)$ is a well-defined Banach space.

We denote $i : L^q \to L^p + L^r; x \mapsto x$ be an inclusion map. Since $1 < p < q < r < \infty$, By Proposition 6.1.3 , we know this map is well-defined, since for each $f \in L^q$, $f \in L^p + L^r$. There is no doubt that i is linear. By the previous discussion, we know $L^p + L^r$ is a normed space and L^q is a normed space by the proof in page 175. Now by Proposition 5.2.2, it is equivalent to prove that i is bounded. We choose $f \in A$ randomly. By Definition of characteristic functions, we have

$$f = f\chi_E + f\chi_{E^c}$$

where we denote $E := \{x \in X : 1 \leq |f(x)| < \|f\|_q\}$.

We prove $f\chi_E \in L^p$.

We have

$$\|f\chi_E\|_p^p = \int_X \left|f\chi_E\right|^p d\mu = \int_E \left|f\right|^p d\mu \leq \int_E \|f\|_q^p d\mu = \|f\|_q^p \mu(E) < \|f\|_q^p < \infty$$

where the first inequality is due to Definition of L^p, the third inequality is due to $f \in L^q$ and the fact we are only interested in $\sigma-$finite measure space and the fifth inequality is due to $\mu(E) < 1$.

We prove $f\chi_{E^c} \in L^r$. We have, since $r > q > 1$

$$\|f\chi_{E^c}\|_r \leq \|f\chi_{E^c}\|_q \leq \|f\|_q < \infty$$

where the second equality is due to properties of integration and the third is due to $f \in L^q$. The technical details involved in the second inequality are

$$\|f\chi_{E^c}\|_q^q = \int_X \left|f\chi_{E^c}\right|^q d\mu = \int_{E^c} \left|f\right|^q d\mu \leq \int_{E^c} \left|f\right|^q d\mu + \int_E \left|f\right|^q d\mu = \int_X \left|f\right|^q d\mu = \|f\|_q^q$$

We prove $\|f\chi_E\|_p \leq \|f\|_q$ and $\|f\chi_{E^c}\|_r \leq \|f\|_q$. From the previous question, we only need to prove that $\|f\chi_E\|_p \leq \|f\|_q$. But we have $\|f\chi_E\|_p^p < \|f\|_q^p$, which immediately implies that $\|f\chi_E\|_p \leq \|f\|_q$

Now we have by Definition of $(A, \|\cdot\|)$

$$\|f\| \leq \max(\|f\chi_E\|_p, \|f\chi_{E^c}\|_r) \leq \|f\|_q$$

where the first inequality is due to Definition of $(A, \|\cdot\|)$, the second is due to properties of integration, and the last is due to

Since such $f \in A$ was chosen randomly, By Definition 5.2,1 where such c can be chosen as 1, we proved that i is bounded and hence we finished the proof.

there is a serious mistake in the last part and we need to use Proposition 6.1.16

□

Question 6.3: Let $(X, \mathcal{M}, \mu)$ be a measure space, and $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$. Prove that $fg \in L^{\frac{pq}{p+q}}$ for any $f \in L^q$ and $g \in L^p$.

Proof. We denote $\hat{p} = \frac{p+q}{p}$ and $\hat{q} = \frac{p+q}{q}$. There is no doubt that $\infty > \hat{p} > 1$ and $\frac{1}{\hat{p}} + \frac{1}{\hat{q}} = \frac{p}{p+q} + \frac{q}{p+q} = 1$. We denote $r := \frac{pq}{p+q}$ for some $1 < r < \infty$. We denote $\hat{f} := |f|^r$ and $\hat{g} = |g|^r$. Since f, g are measurable functions, $\hat{f}, \hat{g}$ are measurable functions. Then applying Holder inequality to $\hat{p}, \hat{q}$ and $\hat{f}, \hat{g}$, we have

$$\|\hat{f}\hat{g}\|_1 \leq \|\hat{f}\|_{\hat{p}} \|\hat{g}\|_{\hat{q}}$$

which is equivalent to Definition 6.1.1(a), we have

$$\int_X \left|\hat{f}\hat{g}\right| \leq \left(\int_X \left|\hat{f}\right|^{\hat{p}}\right)^{\frac{1}{\hat{p}}} \left(\int_X \left|\hat{g}\right|^{\hat{q}}\right)^{\frac{1}{\hat{q}}}. \tag{2}$$

After plugging $\hat{f} = |f|^r$ and $\hat{g} = |g|^r$ into (2), we have

$$\int_X \left|fg\right|^r \leq \left(\int_X \left|f\right|^{r\hat{p}}\right)^{\frac{1}{\hat{p}}} \left(\int_X \left|g\right|^{r\hat{q}}\right)^{\frac{1}{\hat{q}}} \tag{3}$$

After plugging $\hat{p} = \frac{p+q}{p}$, $\hat{q} = \frac{p+q}{q}$ and $r = \frac{pq}{p+q}$ into (2), we have

$$\int_X \left|fg\right|^{\frac{pq}{p+q}} \leq \underbrace{\left(\int_X \left|f\right|^{q}\right)^{\frac{1}{\hat{p}}}}_{:=A} \underbrace{\left(\int_X \left|g\right|^{p}\right)^{\frac{1}{\hat{q}}}}_{:=B} \tag{4}$$

Now by Definition of L^p, we have $\|f\|_q < \infty$ which implies by Definition 6.1.1.(a) that

$$\left(\int_X \left|f\right|^{q}\right)^{\frac{1}{q}} < \infty$$

which implies that $A < \infty$ since $\|f\|_q = A^{\frac{\hat{p}}{q}}$. Similarly we have $B < \infty$. Then by (4) and Definition 6.1.1(a), we have

$$\|fg\|_{\frac{pq}{p+q}}^{\frac{pq}{p+q}} = \int_X \left|fg\right|^{\frac{pq}{p+q}} < \infty$$

which immediately implies that $\|fg\|_{\frac{pq}{p+q}} < \infty$. Then by Definition 6.1.1(b), we have $fg \in L^{\frac{pq}{p+q}}$.

□

Question 6.6: Let $f \in L^p \cap L^\infty$. Prove that

a. $f \in L^q$ for all $q \geq p$,

b. If $\|f\|_\infty > 0$, then for any $\|f\|_\infty > \epsilon > 0$, the set E_ϵ defined by

$$E_\epsilon = \{x : |f(x)| > \|f\|_\infty - \epsilon\}$$

has the properties that $\mu(E_\epsilon) > 0$ and

$$(\|f\|_\infty - \epsilon)^q \chi_{E_\epsilon} \leq |f|^q \leq |f|^p \|f\|_\infty^{q-p} \text{ a.e.}$$

c. $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$

Proof. a. We choose $q \in \mathbb{R}^+$ with $q \geq p$. When $g = p$, $f \in L^p \cap L^\infty$ implies that $f \in L^p = L^q$ and we are done. We consider the case $g > p$. Since $0 < p < q$, by Proposition 6.1.14, we have

$$\|f\|_q \leq \|f\|_p^{\frac{p}{q}} \|f\|_\infty^{1-\frac{p}{q}} \text{a. e.} \tag{5}$$

Since $f \in L^p \cap L^\infty$ implies that $f \in L^p$ and $f \in L^\infty$, by Definition 6.1.1(b), we have

$$\|f\|_p < \infty \text{ and } \|f\|_\infty < \infty \text{ a.e.} \tag{6}$$

Since $(\mathbb{R}, \cdot)$ is a group, by (5) and (6), we have $\|f\|_q < \infty$ a.e., which immediately implies that $f \in L^q$ by Definition 6.1.1(b). Since such q was chosen randomly, we finished the proof.

b. We choose $\epsilon \in (0, \|f\|_\infty)$ randomly. We prove $\mu(E_\epsilon) > 0$ by contradiction. By Definition of measures, we have $\mu(E_\epsilon) \geq 0$. Then we have $\mu(E_\epsilon) = 0$. Since $\|f\|_\infty - \epsilon > 0$ by Definition 6.1.10(a), we have

$$\|f\|_\infty - \epsilon \in M(f) \text{ and } M(f) \neq \varnothing$$

and hence by Definition of inf, we have $\|f\|_\infty \leq \|f\|_\infty - \epsilon$ which implies that $\epsilon \geq 0$, which obviously contradicts with our choice of ϵ. We denote

$$A := \left\{x \in X : (\|f\|_\infty - \epsilon)^q \chi_{E_\epsilon}(x) \leq |f(x)|^q \leq |f(x)|^p \|f\|_\infty^{q-p}\right\}$$

Then by Definition, we are required to prove that $\mu(A^c) = 0$. By Definition 6.1.1(b), we have $\mu(B^c) = 0$, where

$$B = \left\{x \in X : |f(x)|^q \leq |f(x)|^p \|f\|_\infty^{q-p}\right\} = \left\{x \in X : |f(x)|^{q-p} \leq \|f\|_\infty^{q-p}\right\}$$

Now for $x \in E_\epsilon$, the inequalities becomes

$$(\|f\|_\infty - \epsilon)^q \leq |f(x)|^q \leq |f(x)|^p \|f\|_\infty^{q-p}$$

Then it holds by Definition of E_ϵ if $x \in B$. Now for $x \in X \setminus E_\epsilon$, the inequalities becomes

$$0 \leq |f(x)|^q \leq |f(x)|^p \|f\|_\infty^{q-p}.$$

Then it holds by Definition of E_ϵ if $x \in B$. So far we proved that $B = (E_\epsilon \cap B) \cup (E_\epsilon^c \cap B) \subset A$. So $A^c \subset B^c$ which implies that $\mu(A^c) \leq \mu(B^c) = 0$ which finish the proof quickly by positivity of measures.

c. We integrate the inequality in (b) to get

$$\left(\|f\|_\infty - \epsilon\right)^q \mu(E_\epsilon) = \left(\|f\|_\infty - \epsilon\right)^q \int_X \chi_{E_\epsilon} = \int_X \left(\|f\|_\infty - \epsilon\right)^q \chi_{E_\epsilon} \leq \int_X \left|f\right|^q \leq \int_X \left|f\right|^p \|f\|_\infty^{q-p} = \|f\|_\infty^{q-p} \int_X \left|f\right|^p$$

But such $\epsilon > 0$ can randomly small. We have

$$\|f\|_\infty^q \mu(E) \leq \int_X \left|f\right|^q = \|f\|_\infty^{q-p} \int_X \left|f\right|^p$$

which implies that

$$\|f\|_\infty \mu(E)^{\frac{1}{q}} \le \|f\|_q \le \|f\|_\infty^{\frac{q-p}{q}} \|f\|_p^{\frac{p}{q}}.$$

Now after pushing the first inequality into ∞, we have

$$\|f\|_\infty \le \liminf_{q\to\infty} \|f\|_q \tag{7}$$

since $\liminf\limits_{q\to\infty} \mu(E)^{\frac{1}{q}} = 1$ and from the second inequality, we have since $p \le q$, by Proposition 6.1.16

$$\|f\|_q \le \|f\|_\infty^{\frac{q-p}{q}} \|f\|_p^{\frac{p}{q}} \le \|f\|_\infty \mu(X)^{\frac{1}{q}}$$

and after pushing both sides into ∞, we have

$$\limsup_{n\to\infty} \|f\|_q \le \|f\|_\infty \tag{8}$$

since $\limsup\limits_{q\to\infty} \mu(X)^{\frac{1}{q}} = 1$. Now by Definition of lim, combing (7) and (8), we prove that $\|f\|_\infty = \lim\limits_{q\to\infty} \|f\|_q$.

$\square$

Question 6.7: Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $1 \leq p < \infty$ and let f_n and f be measurable functions. Prove that

a. If $f_n \to f$ in L^p, then $f_n \to f$ in measure, and

b. If $f_n \to f$ in measure and there exists a $g \in L^p$ such that $|f_n| \leq g$ for all n, then $f_n \to f$ in L^p.

Proof. (a) We choose $\epsilon > 0$ randomly. We denote $A := \Big\{x \in X : |f_n(x) - f(x)| \geq \epsilon\Big\}$. Since $f_n \to f$ in L^p, by Definition 6.1.1. and continuity of product functions,

$$\int_X \Big|f_n(x) - f(x)\Big|^p d\mu \to \infty \tag{9}$$

But now we have

$$\epsilon^p \mu(\Big\{x \in X : |f_n(x) - f(x)| \geq \epsilon\Big\}) = \epsilon^p \int_A 1 d\mu = \int_A \epsilon^p d\mu \leq \int_A \Big|f_n(x) - f(x)\Big|^p d\mu$$

. So by (9), we have

$$\epsilon^p \lim_{n\to\infty} \mu(\Big\{x \in X : |f_n(x) - f(x)| \geq \epsilon\Big\}) = 0,$$

which immediately implies that

$$\lim_{n\to\infty} \mu(\Big\{x \in X : |f_n(x) - f(x)| \geq \epsilon\Big\}) = 0.$$

Since such $\epsilon > 0$ was chosen randomly, by Definition 2.5.1, we proved that $f_n \to f$ in measure.

(b) By Definition 6.1.1(a) and the continuity of product functions, it is enough to prove that

$$\int_X \Big|f_n(x) - f(x)\Big|^p d\mu \to 0$$

We denote $h_n(x) := \Big(f_n(x) - f(x)\Big)^p$ for each $n \in \mathbb{N}$. Then by Definition 6.1.1(a), it is equivalent to prove that $h_n \to 0$ in L^1. Now for each $n \in \mathbb{N}$,

$$|h_n(x)| = \Big|f_n(x) - f(x)\Big|^p = \Big|f_n(x) + (-f(x))\Big|^p \leq \Big(|f_n(x)| + |f(x)|\Big)^p \leq \Big(g(x) + |f(x)|\Big)^p$$

And we denote $\hat{g}(x) := \Big(g(x) + |f(x)|\Big)^p$. Now by Definition 6.1.1(a), we have

$$\|\hat{g}\|_1 = \int_X \Big|\hat{g}(x)\Big| d\mu = \int_X \Big|g(x) + |f(x)|\Big|^p d\mu = \|g + |f|\|_p^p \leq \Big(\|g\|_p + \|f\|_p\Big)^p \tag{10}$$

Since $f_n \to f$ in measure, by Theorem 2.5.3, we can choose a subsequence f_{n_k} such that $f_{n_k} \to f$ pointwise a.e. We denote

$$A := \Big\{x \in X : \lim_{k\to\infty} f_{n_k}(x) = f(x)\Big\}.$$

Then by Definition we have $\mu(A^c) = 0$. By Definition 6.1.1(a), we have

$$\|f\|_p^p = \int_X \Big|f\Big|^p = \int_A \Big|f(x)\Big|^p d\mu(x) = \int_A \Big|\lim_{k\to\infty} f_{n_k}(x)\Big|^p d\mu(x) = \lim_{k\to\infty} \int_A \Big|f_{n_k}(x)\Big|^p d\mu(x) < \infty$$

since $\mu(X) < \infty$, which implies that $\|f\|_p < \infty$ .Now since $g \in L^p$,by Definition 6.1.1(a), we have $\|g\|_p < \infty$

Then by (10), We have $\hat{g} \in L^1$. We prove that $h_n \to 0$ in measure. We choose $\epsilon > 0$ randomly. We have by Definition 2.5.1 where $f_n \to f$ in measure

$$\lim_{n\to\infty} \mu(\Big\{x \in X : |h_n(x)| \geq \epsilon\Big\}) = \lim_{n\to\infty} \mu(\Big\{x \in X : \Big|f_n(x) - f(x)\Big|^p \geq \epsilon\Big\}) = \lim_{n\to\infty} \mu(\Big\{x \in X : |f_n(x) - f(x)| \geq \epsilon^{\frac{1}{p}}\Big\}) = 0$$

Since such $\epsilon > 0$ was chosen randomly, by Definition 2.5.1, we proved that $h_n \to 0$ in measure. Now since $h_n \to 0$ in measure, $|h_n| \leq \hat{g}$ for each $n \in \mathbb{N}$ and $\hat{g} \in L^1$, by Dominated Convergence Theorem for convergence in measure, we have $h_n \to 0$ in L^1. □

Method 2

Proof. Since $f_n \to f$ in measure, by Theorem 2.5.3, we can choose a subsequence f_{n_k} such that $f_{n_k} \to f$ pointwise a.e.. Now by Definition of subsequence, we have $|f_{n_k}| < g$ for each $k \in \mathbb{N}$. So by the Dominated Convergence for L^1, we have

$$\int_X \left| f_{n_k} - f \right| d\mu \to 0$$

Then we have $|f_{n_k} - f| \to 0$ a.e.. So we can assume that $|f_{n_k} - f| < 1$ a.e.. This implies that

$$0 \leq \int_X \left| f_{n_k} - f \right|^p d\mu \leq \int_X \left| f_{n_k} - f \right| d\mu.$$

Then after pushing both sides into ∞, by (11) and Definition 6.1.1(a), we have

$$\|f_{n_k} - f\|_p \to \infty$$

Now we want to prove this by contradiction and suppose that $\|f_n - f\|_p \nrightarrow 0$, we could construct a subsequence of f_n, $h_i = f_{n_i}$ such that

$$\|h_i - f\|_p \geq \epsilon \tag{11}$$

for some $\epsilon > 0$. We would still have $|h_i| < g$ and $h_i \to f$ in measure. So we may construct subsequence h_{i_k} of h_i such that

$$\|h_{i_k} - f\|_p \to 0. \tag{12}$$

Now (11) and (12) gives us a contradiction. □

Question 6.8: Let $1 \le p < \infty$, $f_n, f \in L^p$, and $f_n \to f$ a.e. prove that $f_n \to f$ in L^p if and only if $\|f_n\|_p \to \|f\|_p$. (Hint: The generalization dominated convergence theorem.)

Proof. We prove if $f_n \to f$ in L^p, then $\|f_n\|_p \to \|f\|_p$ in $\mathbb{R}$. We have that

$$\lim_{n\to\infty} \left| \|f_n\|_p - \|f\|_p \right| \le \lim_{n\to\infty} \|f_n - f\|_p = 0$$

by triangle inequalities of norms, which implies that by Definition of $(\mathbb{R}, |\cdot|)$, $\|f_n\|_p \to \|f\|_p$ in $\mathbb{R}$.
We prove if $\|f_n\|_p \to \|f\|_p$, then $f_n \to f$ in L^p. Since $p \ge 1$, we have inequalities

$$|f_n - f|^p \le (|f_n| + |f|)^p \le (2\max(|f_n(x)|, |f(x)|))^p \le 2^p(|f_n|^p + |f|^p).$$

Then after integrating them on X, we have

$$\int_X \left| f_n - f \right|^p d\mu \le 2^p \int_X \left(\left| f_n \right|^p + \left| g \right|^p \right)$$

which implies that

$$I := \lim_{n\to\infty} \int_X \left| f_n - f \right|^p d\mu \le 2^p \underbrace{\lim_{n\to\infty} \int_X \left| f_n \right|^p}_{:=A} + 2^p \int_X \left| f \right|^p$$

Now since $f_n, f \in L^p$ implies that $|f_n|^p, |f|^p \in L^1$ and $|f_n|^p \to |f|^p$ in L^1, by the generalization dominated convergence theorem, we have

$$A = \int_X \left| f \right|^p = \|f\|_p^p < \infty$$

since $f \in L^p$ and Definition 6.1.1.(b). So

$$I_n \le 2^p A + 2^p A < \infty$$

Then again by Dominated convergence theorem, we have by the continuity of norms and $f_n \to f$ a.e.

$$I = \int_X \lim_{n\to\infty} \left| f_n - f \right|^p d\mu = \int_X \left| \lim_{n\to\infty} f_n(x) - f(x) \right|^p d\mu(x) = 0$$

which says that by Definition 6.1.1(a), $\lim_{n\to\infty} \|f_n - f\|_p^p = 0$ which implies that $\|f_n - f\|_p \to 0$. This proves $f_n \to f$ in L^p by Definition 6.1.1(b). $\square$

Homework11 — Math 5323

Due: Fridays, 15 April 2021, by 11:59 p.m. CDT

Question 6.9: Prove that if $\dim(L^p) > 1$, then L^p norm is a norm induced by an inner product if and only if $p = 2$.(Hint: The parallelogram law.)

Proof. First we proved that if $p = 2$, then $\|\cdot\|_p$ is induced by an inner product. We define

$$\langle \cdot, \cdot \rangle : L^p \times L^p \to \mathbb{C}; (f,g) \mapsto \int f\overline{g} d\mu$$

There is no doubt that this is a well-defined map and we want to check the following items to confirm this is a well-defined inner product according to Definition 5.3.1

i) For $f, g, h \in L^p$ and $a, b \in \mathbb{C}$, by linearity of integrations, we have

$$\langle af + bg, h \rangle = \int \Big(af + bg\Big)\overline{h} d\mu = a \int f\overline{h} d\mu + b \int g\overline{h} d\mu = a\langle f, h\rangle + b\langle g, h\rangle.$$

ii) For any $f, g \in L^p$, $\langle f, g\rangle = \displaystyle\int f\overline{g} d\mu = \overline{\int \overline{f\overline{g}} d\mu} = \overline{\int g\overline{f} d\mu} = \overline{\langle g, f\rangle}$.

iii) For $f \in L^p$, $\langle f, f\rangle = \displaystyle\int f\overline{f} d\mu = \int \Big|f\Big|^2 d\mu \geq 0$. For any $f \in L^p$, $\langle f, f\rangle = \int \Big|f\Big|^2 d\mu = 0$ if and only if $f = 0$ a.e.

Furthermore due to Definition 6.1.1(a), since $p = 2$ we have for any $f \in L^p$

$$\sqrt{\langle f, f\rangle} = \sqrt{\int f\overline{f} d\mu} = \sqrt{\int \Big|f\Big|^2 d\mu} = \|f\|_p$$

which says that $\|\cdot\|_p$ is induced by a norm due to Definition 5.3.4.

Second we prove that if $\|\cdot\|$ is induced by an inner product, then $p = 2$. We prove this by contradiction and hence suppose $p \neq 2$. Then there are two cases. We consider the case $p = \infty$. Since the norm of L^p is induced by an inner product, by Definition 5.3.1, we know L^p is an inner product space. We choose $A, B \in \mathcal{M}$ such that $A \cap B = \varnothing$, $\mu(A) \neq \varnothing$ and $\mu(B) \neq \varnothing$. Then we have by Definition 6.1.10

$$\begin{aligned}
&\|1_A + 1_B\|_\infty^2 + \|1_A - 1_B\|_\infty^2 \\
&= \inf\Big\{\alpha \geq 0 : \mu(\{x \in X : |1_A(x) + 1_B(x)| > \alpha\}) = 0\Big\} + \inf\Big\{\alpha \geq 0 : \mu(\{x \in X : |1_A(x) - 1_B(x)| > \alpha\}) = 0\Big\} \\
&= 1^2 + 1^2 = 2
\end{aligned}$$

and we also have by Definition 6.1.10

$$\begin{aligned}
&2\Big(\|1_A\|_\infty^2 + \|1_B\|_\infty^2\Big) \\
&= 2\Big(\inf\Big\{\alpha \geq 0 : \mu(\{x \in X : |1_A(x)| > \alpha\}) = 0\Big\} + \inf\Big\{\alpha \geq 0 : \mu(\{x \in X : |1_A(x)| > \alpha\}) = 0\Big\}\Big) \\
&= 2(1^2 + 1^2) = 4,
\end{aligned}$$

which gives us a contradiction with the parallelogram rule since L^p is an inner product space.
We consider the case $p < \infty$. There is no doubt that L^p is an inner product space. We choose $A, B \in \mathcal{M}$ with $A \cap B \neq \varnothing$ such that $0 < \mu(A), \mu(B) < \infty$. We denote

$$f_p := \frac{1}{(\mu(A))^{\frac{1}{p}}} 1_A \geq 0 \text{ and } g_p := \frac{1}{(\mu(B))^{\frac{1}{p}}} 1_B \geq 0.$$

Then by Definition 6.1.10, we have

$$2\Big(\|f_p\|_{L^p}^2 + \|g_p\|_{L^p}^2\Big) = 2\Big(\int_X |f_p|^p + \int_X |g_p|^p\Big) = 2(1+1) = 4$$

we have

$$\|f_p + g_p\|_{L^p}^2 + \|f_p - g_p\|_{L^p}^2 = \int_X \left| \frac{1_A}{(\mu(A))^{\frac{1}{p}}} + \frac{1_B}{(\mu(B))^{\frac{1}{p}}} \right|^p d\mu + \int_X \left| \frac{1_A}{(\mu(A))^{\frac{1}{p}}} - \frac{1_B}{(\mu(B))^{\frac{1}{p}}} \right|^p d\mu = 2 \cdot 2^{\frac{2}{p}}$$

and we have $p \neq 2$, which gives a contradiction with the parallelogram rule since L^p is an inner product space.

$\square$

Question 6.10: Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $k(x,t) \in L^q(\mathbb{R}^2, m^2)$. Prove that for any $f(t) \in L^p(\mathbb{R}, m(t))$, $k(x,t)f(t) \in L^1(\mathbb{R}, m(t))$ for a.e. x. and

$$T(f)(x) = \int_{\mathbb{R}} k(x,t)f(t)dt$$

is a bounded linear operator from $L^p(\mathbb{R}, m)$ to $L^q(\mathbb{R}, m)$ with $\|T\| \leq \|k\|_q$

Proof. We choose $f(t) \in L^P(\mathbb{R}, m)$ and $k(x,t)f(t) \in L^1(\mathbb{R}, m(t))$ for a.e. x. randomly. Since L^p denotes the quotient space by Definition 6.1.1(b), if the statement involved in x is true , then it must be true a.e., and for the simlicity, we dont use the word *a.e* to emphasize this, which is to say elements of L^p denotes the equivalence class and precisely f denote $[f]$ for each $f \in L^p$.
First we want to prove that T is linear. For any $\alpha \in \mathbb{C}$, $f, g \in L^p(\mathbb{R}, m(t))$ and any $x \in \mathbb{R}$, we have by linearity of integration

$$T(\alpha f + g)(x) = \int_{\mathbb{R}} k(x,t)(\alpha f + g)(t)dt = \alpha \int_{\mathbb{R}} k(x,t)f(t)dt + \int_{\mathbb{R}} k(x,t)g(t)dt = \alpha T(f)(x) + T(g)(x).$$

Second we want to prove that T is well-defined. We choose $f \in L^p(\mathbb{R}, m(t))$ randomly. We have

$$\begin{aligned} I := \int_{\mathbb{R}} \Big|T(f)(x)\Big|^q d\mu(x) &= \int_{\mathbb{R}} \Big|\int_{\mathbb{R}} k(x,t)f(t)dt\Big|^q d\mu(x) && (1)\\ &= \int_{\mathbb{R}} \left(\Big|\int_{\mathbb{R}} k(x,t)f(t)dt\Big|\right)^q d\mu(x) && (2)\\ &\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Big|k(x,t)f(t)\Big|dt\right)^q d\mu(x) && (3)\\ &= \int_{\mathbb{R}} \left(\|k(x,\cdot)f\|_1\right)^q d\mu(x), && (4) \end{aligned}$$

where (3) is by triangle inequalities and (4) is by Definition 6.1.1(a). By Definition 6.1.1(b) and (a), $k(x,t) \in L^q(\mathbb{R}^2, m^2)$ implies that by Theorem 2.6.13

$$\infty > \int_{\mathbb{R}\times\mathbb{R}} \Big|k(x,t)\Big|^q d(m \times m)(x,t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \Big|k(x,t)\Big|^q dm(t)dm(x) \tag{5}$$

which implies that

$$\infty > \int_{\mathbb{R}} \Big|k(x,t)\Big|^q dm(t)$$

which immediately implies that by Definition 6.1.1(b), $k(x,\cdot) \in L^q(\mathbb{R}, m)$. Furthermore, since $f \in L^p(\mathbb{R}, m)$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, by Holder's inequality, we have

$$\|fk(x,\cdot)\|_1 \leq \|k(x,\cdot)\|_q \|f\|_p \tag{6}$$

Combing (4) and (6), we have

$$\begin{aligned} I &\leq \int_{\mathbb{R}} \left(\|k(x,\cdot)\|_q \|f\|_p\right)^q d\mu(x) && (7)\\ &= \|f\|_p^q \int_{\mathbb{R}} \|k(x,\cdot)\|_q^q d\mu(x) && (8)\\ &= \|f\|_p^q \int_{\mathbb{R}} \int_{\mathbb{R}} \Big|k(x,t)\Big|^q dm(t)dm(x) && (9)\\ &< \infty && (10) \end{aligned}$$

where (8) is by properties of integration, (9) is by Definition 6.1.1(a) and (10) is by (5) and Definition 6.1.1(b) with $f \in L^p(\mathbb{R}, m)$. So by Definition 6.1.1(a), $\|T(f)\|_q = I^{\frac{1}{q}} < \infty$. But such $f \in L^p(\mathbb{R}, m)$ is chosen randomly. We proved that T is well-defined.
Third, we prove that T is bounded with $\|T\| \leq \|k\|_q$. By Definition 5.2.3 and Definition 6.1.1(b), since $k \in L^q(\mathbb{R}^2, m^2)$, it is enough to prove that $\|T\| \leq \|k\|_q$. We choose $f(t) \in L^p(\mathbb{R}, m(t))$ with $\|f\|_p = 1$ randomly. By Definition 5.2.3, we are required to prove that $\|T(f)\|_q \leq \|k\|_q$. But by Definition 6.1.1(a), (9) and our choice of f, we have

$$\|T(f)\|_q = I^{\frac{1}{q}} \leq \|f\|_p \|k\|_q = \|f\|_p$$

□

Comment: In this question, μ and m are exchangeable.

Question 6.11: Show that $L^p(\mathbb{R}^n, m^n)$ is separable for all $1 \le p < \infty$, and $L^\infty(\mathbb{R}^n, m^n)$ is not separable.

Proof. First we show $L^p(\mathbb{R}^n, m^n)$ is separable for all $1 \le p < \infty$. We choose $p \in [1, \infty)$ randomly.By Definition 4.1.3(m), we need to find a countable subset S of $L^p(\mathbb{R}^n, m^n)$ such that $\overline{S} = L^p(\mathbb{R}^n, m^n)$. We denote $C_c(\mathbb{R}^n)$ be the continuous functions $\mathbb{R}^n \to \mathbb{R}$ with compact support. Since $1 \le p < \infty$, by Theorem 6.1.8, we know that $\overline{C_c(\mathbb{R}^n)} = L^p(\mathbb{R}^n, m^n)$. We denote $\mathcal{P}$ be the set of polynomials with real coefficients defined on $\mathbb{R}^n$ having compact supports. By Stone–Weierstrass theorem, we know that $\overline{\mathcal{P}} = C_c(\mathbb{R}^n)$. Since the closure of a closed set is itself, now it is enough to find a subset $A \subset \mathcal{P}$ such that $\overline{A} = \mathcal{P}$. We denote A be the set of polynomials with rational coefficients defined on $\mathbb{R}^n$. To prove it is countable, by Definition of countable in the knowledge of the set theory, since the countable product of countable sets is still countable and $\mathbb{Q}$ is countable, it is enough to find a bijection map from A to $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \cdots$. By Definition of polynomials functions, we have

$$A = \left\{ \mathbb{R}^n \ni f : x \mapsto a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{R} : n \in \mathbb{N}_0, a_0, \cdots a_n \in \mathbb{Q} \ f \text{ has a compact support} \right\} \tag{11}$$

We define $\alpha : A \to \mathbb{Q} \times \mathbb{Q} \times \cdots$ by $f \mapsto (a_0, a_1, a_2, \cdots, a_n, 0, \cdots)$. There is no doubt that α is well-defined map by (11). It is obviously injective and it is surjective since $\mathbb{Q} \times \mathbb{Q} \times \cdots$ actually denotes the set collecting a sequence in $\mathbb{Q}$ of finite non zero terms. We prove that A is dense in $\mathcal{P}$. We choose $\epsilon > 0$ and $f \in \mathcal{P}$ randomly. By Definition of topology and the fact that the topology on normed space is induced by its norm, we are required to find $g \in A$ such that $\|g - f\|_p < \epsilon$. We denote $n := deg(p)$ for some $n \in \mathbb{R}$. There is nothing to prove for $n = 0$ since $\mathbb{Q} \subset A$ and $\overline{\mathbb{Q}} = \mathbb{R}$ and all norms on finite dimensional real vector space induce the same topology. For $n \ge 1$, we write $f = f_0 + f_1 + \cdots + f_n$ where we denote $f_k := a_k x^k$ for each $k = 0, \cdots, n$. Now since by triangle inequality of norms, we have

$$\|g - f\|_p \le \sum_{k=0}^{n} \|g_i - f_i\|_p$$

where we denote $g := g_0 + \cdots + g_n$ and $g_k := b_k x^k$ for each $0 \le k \le n$. It is enough to prove that for each $0 \le k \le n$, we can find $g_i = b_i x^i$ such that $\|g_i - f_i\|_p < \frac{\epsilon}{n+1}$ by Definition 6.1.1(a), which is equivalent to

$$\int_K \left| b_i x^i - a_i x^i \right|^p d\mu(x) = \left| a_i - b_i \right|^p \int_K |x^i|^p dm(x) < \left(\frac{\epsilon}{n+1} \right)^p$$

where we denote $K \subset \mathbb{R}^n$ denote the compact support of f, due to $\mu(K) < \infty$ by Definition of m which is implied by

$$\left| a_i - b_i \right| < \left(\frac{1}{\mu(K)} \right)^{\frac{1}{p}} \left(\frac{1}{M} \right)^{\frac{1}{p}} \frac{\epsilon}{n+1}$$

where we denote $M := \sup_{x \in K} \left| x \right|^{ip}$ for some $M \in \mathbb{R}$ by the fact continuous functions has extreme values on compact sets. This is true, by Definition of normed spaces, since $\mathbb{Q}$ is dense in the normed space $(\mathbb{R}, |\cdot|)$.

To avoid those dark analysis technical details, maybe we can argue A is dense in $\mathcal{P}$ in this way. We know that

$$\mathcal{P} = \bigoplus_{k=1}^{\infty} \mathbb{R}[x^k] \text{ and } A = \bigoplus_{k=1}^{\infty} \mathbb{Q}[x^k]$$

by Definition of the product of topology, it is enough to prove that for each $k = 0, 1, \cdots$, the subspace $\mathbb{Q}[x^k]$ is dense in the space $\mathbb{R}[x^k]$. Also it is easy to check that $\mathbb{R}[x^k] := \{\alpha x^k : \alpha \in \mathbb{R}\} = \{\alpha e^{(k)} : \alpha \in \mathbb{R}\}$ is a 1-dimensional real vector space where we denote $e^{(k)} := x^k$ and the vector addition is defined as $\alpha e^{(k)} - \beta e^{(k)} := (\alpha - \beta) e^{(k)}$. Since any well-defined norms on a finite dimensional vector space induces the same topology, as long as we construct a norm $\|\cdot\|$ on $\mathbb{R}[x^k]$ such that

$$\text{for any } f \in \mathbb{R}[x^k] \text{ any } \epsilon > 0, \text{ there exists } g \in \mathbb{Q}[x^k] \text{ satisfy } \|g - f\| < \epsilon (*).$$

But now we define a norm $\|\cdot\| : \mathbb{R}[x^k] \to [0, \infty); \alpha e^{(k)} \mapsto |\alpha|$ where $|\cdot|$ denotes the absolute function and check it is well-defined norm and prove it satisfy (*) by using the fact $\mathbb{Q}$ is dense in $(\mathbb{R}, |\cdot|)$. We are done.

To make logic more neat, maybe we argue $\mathbb{Q}[x^k]$ is dense in $\mathbb{R}[x^k]$ for each i in this way. Since any norms defined on a finite dimensional vector space induce the same topology, topology isomorphism per serves the topological properties and $\mathbb{Q}$ is dense in the topology $(\mathbb{R}, |\cdot|)$, it is enough to construct an isomorphism f from $\mathbb{R}$ to $\mathbb{R}[x^k]$ such that $f(\mathbb{Q}) = \mathbb{Q}[x^k]$. We consider $f : \alpha \mapsto \alpha x^k$. There is no doubt this is well-defined linear bijective map and $f(\mathbb{Q}) = \mathbb{Q}[x^k]$. Now to use the Corollary 5.2.17 to prove f is an isomorphism, we need to construct inner products on $\mathbb{R}[x^k]$ and $\mathbb{R}$ respectively, so that $\mathbb{R}[x^k]$ and $\mathbb{R}$ are both complete with respect to norms induced by inner products and the operator norm of f is finite with respect to these norms induced by those inner products. Now we define an inner product $\langle \alpha x^k, \beta x^k \rangle := \alpha\beta$ and use the usual product on $\mathbb{R}$. All remaining checking works are left for readers and the completeness of $(\mathbb{R}[x^k], \langle \cdot, \cdot \rangle)$ is essentially due to the completeness of $(\mathbb{R}, |\cdot|)$.

Second we show that $L^\infty(\mathbb{R}^n, m^n)$ is not separable. By Definition 4.1.3(m), it is equivalent to find an uncountable set $\mathcal{B}$ such that the distance of any element of $\mathcal{B}$ is greater than 1. We denote

$$\mathcal{B} := \left\{ f_s(x) = \chi_{[0,s]}(x) : 0 \leq s \leq 1 \right\} = \left\{ f_s \right\}_{s \in [0,1]}.$$

There is no doubt that $\mathcal{B}$ is uncountable. since its index is $[0,1]$ which is obviously uncountable. We choose $s, t \in [0,1]$ with $s < t$ randomly and we consider

$$\|f_s - f_t\|_\infty = \inf\{\alpha \geq 0 : \mu(\left\{ x \in X : |f_s(x) - f_t(x)| > \alpha \right\}) = 0\}$$

Now it remains to prove that $\|f_s - f_t\|_\infty \geq 1$. We choose $\alpha \geq 0$ such that $\mu(\left\{ x \in X : |f_s(x) - f_t(x)| > \alpha \right\}) = 0$ randomly. By Definition of inf, it is enough to prove that $\alpha \geq 1$. We argue this by contradiction and suppose that $0 \leq \alpha < 1$. Then we have

$$\mu(\left\{ x \in X : \left| \chi_{[0,s]}(x) - \chi_{[0,t]}(x) \right| = 1 \right\}) = 0 \tag{12}$$

We denote $A := \left\{ x \in X : \left| \chi_{[0,s]}(x) - \chi_{[0,t]}(x) \right| = 1 \right\}$ Since the value of characteristic functions are only 0 or 1 and $s < t$, the only possible value of $\chi_{[0,s]}(x) = 0$ and $\chi_{[0,t]}(x) = 1$. So $A = (s,t)$ and hence $\mu(A) = \mu((s,t)) = t - s > 0$ which contradicts with (12). $\square$

Question 6.12: Let $(X, \mathcal{M}, \mu)$ be a $\sigma-$finite measure space and $g \in L^{\infty}(\mu)$. Prove that the operator defined by $T(f) = fg$ is a bounded linear operator on $L^p(\mu)$ for all $1 \leq p < \infty$, and $\|T\| = \|g\|_{\infty}$.

Proof. We choose $1 \leq p < \infty$ randomly. We want to prove that $T \in L(L^p(\mu), L^p(\mu))$. We prove that T is linear. For $\alpha \in \mathbb{R}$, $f_1, f_2 \in L^p(\mu)$,

$$T(\alpha f_1 + f_2) = \Big(\alpha f_1 + f_2\Big) g = \alpha(f_1 g) + f_2 g = \alpha T(f_1) + T(f_2)$$

We prove that T is well-defined. We choose $f \in L^p(\mu)$ randomly. Now we have

$$\begin{aligned}
\|T(f)\|_p^p &= \int_X \Big|fg\Big|^p d\mu && (13)\\
&= \int_X \Big|f^p g^p\Big| d\mu && (14)\\
&= \|f^p g^p\|_1 && (15)\\
&\leq \|f^p\|_1 \|g^p\|_{\infty} && (16)\\
&= \int_X \Big|f^p\Big| d\mu \|g^p\|_{\infty} && (17)\\
&= \int_X \Big|f\Big|^p d\mu \|g^p\|_{\infty} && (18)\\
&= \|f\|_p^p \|g^p\|_{\infty} && (19)
\end{aligned}$$

where (13) and (15) is due to Definition 6.1.1(a), (16) is due to Theorem 6.1.12(a), (17) and (19) is due to Definition 6.1.1(a) and (20) is due to our choice of f. By Definition 6.1.10(a) and Definition of inf, we have

$$\begin{aligned}
\|g^p\|_{\infty} &= \inf\left\{\alpha \geq 0 : \mu(\left\{x \in X : |g^p(x)| > \alpha\right\}) = 0\right\} && (20)\\
&= \inf\left\{\alpha \geq 0 : \mu(\left\{x \in X : |g(x)| > \alpha^{\frac{1}{p}}\right\}) = 0\right\} && (21)\\
&= \|g\|_{\infty}^p && (22)
\end{aligned}$$

Combing (19) and (22), we have that

$$\|T(f)\|_p^p \leq \|g\|_{\infty}^p \|f\|_p^p < \infty \qquad (23)$$

where the last inequality is due to Definition6.1.1(b) and Definition 6.1.10 with $g \in L^{\infty}(\mu)$ and $f \in L^p(\mu)$ which implies that $\|T(f)\|_p < \infty$ immediately. Then by Definition 6.1.1(b), we proved that T is well-defined since such $f \in L^p(\mu)$ was chosen randomly.

We prove that $\|T\| = \|g\|_{\infty}$ .We choose $f \in L^p(\mu)$ with $\|f\|_p = 1$ randomly. Then by (23), we have

$$\|T(f)\|_p^p \leq \|g\|_{\infty}^p$$

which implies that due Definition 6.1.10 with $g \in L^{\infty}(\mu)$,

$$\|T(f)\|_p \leq \|g\|_{\infty}$$

Then since such f was chosen randomly, by Definition 5.2.3, we proved that $\|T\| \leq \|g\|_{\infty}$. Now it remains to prove that $\|g\|_{\infty} \leq \|T\|$. We choose $0 < \epsilon < \|g\|_{\infty}$ randomly. Then $\|g\|_{\infty} - \epsilon < \|g\|_{\infty}$. Then by Definition 6.1.10 and Definition of inf, we have that $\|g\|_{\infty} - \epsilon \notin \left\{\alpha \geq 0 : \mu(\left\{x \in X : |g(x)| > \alpha\right\}) = 0\right\}$. But we also have $\|g\|_{\infty} - \epsilon > 0$ where $\|g\|_{\infty} > \epsilon$. Then by positivity of measures, we have

$$\mu(\left\{x \in X : |g(x)| > \|g\|_{\infty} - \epsilon\right\}) > 0. \qquad (24)$$

We denote $E_{\epsilon} = \left\{x \in X : |g(x)| > \|g\|_{\infty} - \epsilon\right\}$. Since $(X, \mathcal{M}, \mu)$ is $\sigma-$finite measure space, we can choose $A_{\epsilon} \supset E_{\epsilon}$ such that $\mu(A_{\epsilon}) < \infty$. Also by the monotonicity of measures and (24), we have $\mu(A_{\epsilon}) > 0$. For making life easier, we keep the notation E_ϵ. Now By Definition 6.1.1(b) and $0 < \mu(E_\epsilon) < \infty$, we have

$$\begin{aligned}
\|T(\chi_{E_\epsilon})\|_p^p &= \int_X \left|\chi_{E_\epsilon} g\right|^p d\mu \\
&= \int_{E_\epsilon} \left|\chi_{E_\epsilon} g\right|^p d\mu \\
&\geq \int_{E_\epsilon} \left|\chi_{E_\epsilon}\right|^p \left(\|g\|_\infty - \epsilon\right)^p d\mu \\
&= \left(\|g\|_\infty - \epsilon\right)^p \|\chi_{E_\epsilon}\|_p^p
\end{aligned}$$

which implies that by Definition 5.2.3

$$\|T\| \geq \frac{\|T(\chi_{E_\epsilon})\|_p}{\|\chi_{E_\epsilon}\|_p} \geq \|g\|_\infty - \epsilon$$

But $\epsilon > 0$ could be randomly small. So $\|T\| \geq \|g\|_\infty$. □

Question 6.13: Let $(X, \mathcal{M}, \mu)$ be a measure space, and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that for any bounded linear operator T on $L^p(\mu)$, there is a bounded linear operator T^* on $L^q(\mu)$ with $\|T^*\| = \|T\|$ such that

$$\int_X T(f) g d\mu = \int_X f T^*(g) d\mu (**)$$

for any $f \in L^p(\mu)$ and $g \in L^q(\mu)$.

Proof. We define the rule of the operator $T^* : L^q(\mu) \to L^q(\mu)$ in the following process. We choose $g \in L^q$ randomly. We define a map $\phi : L^p(\mu) \to \mathbb{R}; f \mapsto \int T(f) g d\mu$. This is a well-defined bounded linear operator, i.e. $\phi \in (L^p(\mu))^*$. We check this in the following items

(i) (Well-defined) For $f \in L^p(\mu)$, we have

$$\left|\phi(f)\right| = \left|\int T(f) g d\mu\right| \le \int \left|T(f) g\right| d\mu = \|T(f) g\|_1 \le \|T(f)\|_p \|g\|_q < \infty$$

where the second is due to triangle inequalities, the third is due to Definition 6.1.1(i), the forth is due to Holder's inequality where $T(f) \in L^p(\mu)$ and the last is due to $T(f) \in L^p$ and $g \in L^q$, which immediately implies that $\phi(f) \in \mathbb{R}$.

(ii) (Linearity) For any $\alpha \in \mathbb{R}$, and any $f_1, f_2 \in L^p(\mu)$, by linearity of integration and linearity of T, we have

$$\phi(\alpha f_1 + f_2) = \int_X T(\alpha f_1 + f_2) g d\mu = \alpha \int_X T(f_1) g d\mu + \int_X T(f_2) g d\mu = \alpha\phi(f_1) + \phi(f_2).$$

(iii) (Bounded) For $f \in L^p(\mu)$, we have by Definition 5.2.3 and (i)

$$\left|\phi(f)\right| \le \|T(f)\|_p \|g\|_q \le \|T\| \|f\|_p \|g\|_q = \Big(\|T\| \|g\|_q\Big) \|f\|_p. \tag{25}$$

Now since $T \in L(L^p(\mu), L^p(\mu))$ and $g \in L^q(\mu)$, by Definition 6.1.1(2), we have $\|T\| \|g\|_q < \infty$. Now by Definition 5.2.1, in (25), we can choose $c := \|T\| \|g\|_q$ which is independent of f.

Now since $\phi \in (L^p(\mu))^*$, $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, by Theorem 6.2.4, we have a unique $h \in L^q(\mu)$ such that $\phi = \phi_h$. Now we assign this h to g. We proved that T^* is well-defined. We prove that T^* is linear. We choose $\alpha \in \mathcal{R}$ and $g_1, g_2 \in L^q$ randomly. By the linearity, we are required to prove that

$$h_3 := T^*(\alpha g_1 + g_2) = \alpha T^*(g_1) + T^*(g_2)$$

which is equivalent to by the uniqueness in Theorem 6.2.4,

$$\phi_{h_3} = \phi_{(\alpha h_1 + h_2)}$$

which is equivalent to by Definition of ϕ, for any $f \in L^p(\mu)$

$$\int T(f)(\alpha g_1 + g_2) d\mu = \alpha \int T(f) g_1 d\mu + \int T(f) g_2 d\mu$$

and this is true by linearity of T. We prove that T^* is bounded. For any $g \in L^q$ with $\|g\|_q = 1$, then by proposition 6.2.1 and Definition of T^*, we have

$$\|T^*(g)\|_q = \|h\|_q = \|\phi_h\| = \|\phi\| \le \|T\| \|g\|_q = \|T\|$$

which implies that by Definition 5.2.3

$$\|T^*\| \le \|T\|. \tag{26}$$

By Definition 5.2.3, we proved that T^* is bounded. We prove (**). For any $f \in L^p$ and $g \in L^q$, we have

$$\int_X f T^*(g) d\mu = \int_X f h d\mu = \phi_h(f) = \phi(f) = \int_X T(f) g d\mu. \tag{27}$$

Now we define $\phi^* : L^q \to L^q$ by $\phi^*(g) = \int_X f T^*(g) d\mu$. By the symmetric property in (26), replacing ϕ by ϕ^* in the previous proof, we have

$$\|T\| \le \|T^*\|. \tag{28}$$

Combing (26) and (28), we proved that $\|T\| = \|T^*\|$. ☐

Homework 12 — Math 5323

Due: Fridays, 25 April 2022, by 11:59 p.m. CDT

Question 6.17:

a. Show that $\sin(nx) \to 0$ weakly in $L^2([0,2\pi],m)$, but not a.e. or in measure. (Hint: Exercise 5.28)

b. Show that $n\chi_{(0,\frac{1}{n})} \to 0$a.e. and in measure, but not weakly in $L^p([0,1],m)$ for any $1 \geq p \geq \infty$.

Proof. a. We denote $\mathcal{H} := L^2([0,2\pi],m)$. Since $\mathcal{H}$ is an infinite-dimensional Hilbert space, by Exercise 5.28(a), Definition 6.1.1(a) and Definition 5.3.11, it is enough to prove that for $n,k \in \mathbb{N}$ with $n \neq k$

$$\int_{[0,2\pi]} sin(nx)sin(kx)dx = 0 \text{ and } \int_{[0,2\pi]} sin(nx)sin(nx)dx = 1$$

Now we choose $n,k \in \mathbb{N}$ with $n \neq k$ randomly and we compute with the help of matlab (Let us leave this stupid computation for computers)

$$\int_{[0,2\pi]} sin(nx)sin(kx)dx = 0$$

and

$$\int_{[0,2\pi]} sin(nx)sin(nx)dx = 1$$

But such n,k was chosen randomly. So we proved that $sin(nx) \to 0$ weakly in $\mathcal{H}$. We disprove that $sin(nx) \to 0$ a.e. by contradiction and hence we assume that $sin(nx) \to 0$ .a.e. Then by Definition,

$$\mu(A) = 0$$

where $A = \left\{x \in [0,2\pi] : \lim_{n\to\infty} sin(nx) = 0\right\}$ and we have

$$A = \left\{x \in [0,2\pi] : \lim_{n\to\infty} \sin^2(nx) = 0\right\} \tag{1}$$

Since $|\sin^2(n_i x)| \leq 1$, by DCT, we have by (1) and properties of integration

$$\pi = \lim_{n\to\infty} \int_0^{2\pi} \sin^2(nx)d\mu(x) = \lim_{n\to\infty} \int_{[0,2\pi]} \sin^2(nx)d\mu(x) = \int_{[0,2\pi]} \lim_{n\to\infty} \sin^2(nx)d\mu(x) \leq \int_A \lim_{n\to\infty} \sin^2(nx)d\mu(x) = 0$$

which is obviously a contradiction. So by Theorem 2.5.3, we disprove that $\sin(nx) \to 0$ in measure.

b. We have

$$\mu(A) = \int_A 1d\mu(x)$$

where we denote $A := \left\{x \in [0,1] : \lim_{n\to\infty} n\chi_{(0,\frac{1}{n})}(x) = 0\right\}$. We denote $g := \chi_{[0,1]}$. Then by Riesz Representation Theorem

$$\langle n\chi_{(0,\frac{1}{n})}, g\rangle - \langle n\chi_{(0,\frac{1}{n})}, 0\rangle = \int_{[0,1]} n\chi_{(0,\frac{1}{n})}\chi_{[0,1]}d\mu(x) = \int_{(0,\frac{1}{n})} nd\mu(x) = 1 \nrightarrow 0$$

So by Definition 5.3.11, we proved that $n\chi_{(0,\frac{1}{n})} \nrightarrow 0$ weakly in $L^p([0,1],m)$. We denote

$$\mathcal{A} := \left\{x \in [0,1] : \lim_{n\to 0} n\chi_{(0,\frac{1}{n})}(x) = 0\right\}.$$

details needed to be done

We choose $1 \geq \epsilon > 0$ randomly. We find

$$\lim_{n\to\infty} \mu(\{x \in [0,1] : \left|n\chi_{(0,\frac{1}{n})}(x)\right| \geq \epsilon\}) = \lim_{n\to\infty} \mu((0, \frac{1}{n})) = 0$$

So by Definition 2.5.1, we proved that $n\chi_{(0,\frac{1}{n})} \to 0$ in measure.

$\square$

Question 6.18: Let $k(x,y)$ be Lebesgue measurable function on $(0,\infty)\times(0,\infty)$ such that $k(\lambda x,\lambda y)=\frac{1}{\lambda}k(x,y)$ for all $\lambda>0$ and

$$\int_0^\infty \big|k(z,1)\big|dz<\infty$$

Prove that

$$T(f)(y)=\int_0^\infty k(x,y)f(x)dx$$

is a well-defined bounded linear operator on $L^\infty(0,\infty)$, and

$$\|T\|\leq\int_0^\infty \big|k(z,1)\big|dz$$

Proof. We prove that it is well-defined. We choose $f\in L^\infty(0,\infty)$ randomly. By Definition of maps, we are required to prove that $T(f)\in L^\infty(0,\infty)$. First there is no doubt that $T(f):(0,\infty)\to\mathbb{R}$ is a well defined measurable function since the integration of measurable functions is still measurable and we want to check it is a well-defined map. We have for $f\in L^\infty(0,\infty)$ and $y\in(0,\infty)$

$$\Big|T(f)(y)\Big|\leq\int_0^\infty\Big|k(x,y)f(x)\Big|dx\leq\int_0^\infty|k(z,1)|dz\|f\|_\infty<\infty \text{ for a.e. } y$$

where $z:=xy$ and transformation of variables are left for readers to check using $k(\lambda x,\lambda y)=\frac{1}{\lambda}k(x,y)$ for $\lambda>0$ since $\|f\|_\infty<\infty$ a.e. which implies that $T(f)(y)\in\mathbb{R}$. Then by Definition 6.1.10(a), we have

$$\|T(f)\|_\infty\leq\int_0^\infty|k(z,1)|dz\|f\|_\infty<\infty \tag{2}$$

Now we prove the linearity of T. For any $\alpha\in\mathbb{R}$, any $f_1,f_2\in L^\infty(0,\infty)$ and any $y\in(0,\infty)$, we have

$$T(\alpha f_1+f_2)(y)=\int_0^\infty k(x,y)(\alpha f_1+f_2)(y)dx=\alpha\int_0^\infty k(x,y)f_1(y)dx+\int_0^\infty k(x,y)f_2(y)dx=(\alpha T(f_1)+T(f_2))(y).$$

We prove the boundedness of T. We choose $f\in L^\infty(0,\infty)$ with $\|f\|_\infty=1$. Then (2) implies that

$$\|T(f)\|_\infty\leq\int_0^\infty|k(z,1)|dz.$$

But such f was chosen randomly. By Exercise 5.3, we have

$$\|T\|\leq\int_0^\infty|k(z,1)|dz$$

□

Question 6.19: (Hilbert's inequality) Let $1 < p < \infty$.Prove that the operator

$$T(f)(x) = \int_0^\infty \frac{f(y)}{x+y} dy$$

is a bounded linear operator on $L^p(0,\infty)$, and

$$\|T\| \le \int_0^\infty \frac{1}{z^{\frac{1}{p}}(z+1)} dz$$

Proof. We define $k : (0,\infty) \times (0,\infty) \to (0,\infty)$ by $k(x,y) = \frac{1}{x+y}$. There is no doubt that k is measurable since any continuous measurable function are measurable. Also we have for each $\lambda > 0$, $k(\lambda y, \lambda x) = \frac{1}{\lambda}\frac{1}{x+y} = \frac{1}{\lambda}k(x,y)$ and

$$\int_0^\infty \frac{|k(z,1)|}{z^{\frac{1}{p}}} dz = \int_0^\infty \frac{1}{(z+1)z^{\frac{1}{p}}} dz < \infty$$

which is given by Matlab. Since $1 \le p \le \infty$, applying Theorem 6.3.4, we have

$$T(f)(x) = \int_0^\infty \frac{f(y)}{x+y} dy$$

are well-defined bounded linear operator on $L^p(0,\infty)$ and

$$\|T\| \le \int_0^\infty \frac{|k(z,1)|}{z^{\frac{1}{p}}} dz = \int_0^\infty \frac{1}{(z+1)z^{\frac{1}{p}}} dz$$

$\square$

Question 6.20: Let $1 < p < \infty$, q be conjugate to p, and $k(x), f(x), g(x)$ be positive Lebesgue measurable functions on $(0, \infty)$. Prove that

$$\int_0^\infty \int_0^\infty k(xy)f(x)g(y)dxdy \leq \underbrace{\left(\int_0^\infty \frac{k(x)}{x^{\frac{1}{q}}}dx\right)\left(\int_0^\infty \frac{f(x)^p}{x^{2-p}}dx\right)^{\frac{1}{p}}}_{=:I}\left(\int_0^\infty g(x)^q dx\right)^{\frac{1}{q}}$$

(Hint: Apply the Holder inequality to the y integral, change variable $u = xy$, apply the Minkowski's inequality for integrals, and then change variable $z = \frac{u}{y}$)

Proof. We have that

$$\begin{aligned}
\int_0^\infty \int_0^\infty k(xy)f(x)g(y)dxdy &= \int_0^\infty g(y)\left(\int_0^\infty k(xy)f(x)dx\right)dy && (3)\\
&= \int_0^\infty g(y)F(y)dy && (4)\\
&= \int_0^\infty \Big|g(y)F(y)\Big|dy && (5)\\
&= \|gF\|_1 && (6)\\
&\leq \|g\|_q\|F\|_p && (7)\\
&= \left(\int_0^\infty g(x)^q dx\right)^{\frac{1}{q}}\|F\|_p && (8)
\end{aligned}$$

where we denote $F(y) := \int_0^\infty k(xy)f(x)dx$, (5) is due the positivity of k, f, g, (6) is to Definition 6.1.1(a), (7) is due to Holder's inequality and (8) is due to Definition 6.1.1(a). Now to finish the proof, it remains to prove that $\|F\|_p \leq I$. But actually we have

$$\begin{aligned}
\|F\|_p &= \left(\int_0^\infty \Big|F(y)\Big|^p dy\right)^{\frac{1}{p}} && (9)\\
&= \left(\int_0^\infty \Big|\int_0^\infty k(xy)f(x)dx\Big|^p dy\right)^{\frac{1}{p}} && (10)\\
&= \left(\int_0^\infty \left(\int_0^\infty k(xy)f(x)dx\right)^p dy\right)^{\frac{1}{p}} && (11)\\
&= \left(\int_0^\infty \left(\int_0^\infty \frac{1}{y}k(u)f(\frac{u}{y})du\right)^p dy\right)^{\frac{1}{p}} && (12)\\
&\leq \int_0^\infty k(u)\left(\int_0^\infty \left(\frac{f(\frac{u}{y})}{y}\right)^p dy\right)^{\frac{1}{p}} du && (13)\\
&= \int_0^\infty k(u)\left(\frac{f(z)^p}{u^{p-1}z^{2-p}}\right)^{\frac{1}{p}} dudz && (14)\\
&= \left(\int_0^\infty \frac{k(x)}{x^{\frac{1}{q}}}dx\right)\left(\int_0^\infty \frac{f(x)^p}{x^{2-p}}dx\right)^{\frac{1}{p}} && (15)
\end{aligned}$$

where (9) is due to Definition 6.1.1(a), (11) is due to positivity of F, (12) is due to we denote $u = xy$, (13) is due to Minkowski's inequality, (14) is due to we denote $z = \frac{u}{y}$ and (15) is due to $\frac{1}{p} + \frac{1}{q} = 1$.

$\square$

Question 6.21: Let $k(x)$ be a Lebesgue measurable function on $(0,\infty)$ such that $\displaystyle\int_0^\infty \frac{|k(u)|}{u^{\frac{1}{2}}}du < \infty$. Prove that

$$T(f)(x) = \int_0^\infty k(xy)f(y)dy$$

is a well-defined bounded linear operator on $L^2(0,\infty)$, and $\|T\| \leq \int_0^\infty \frac{|k(u)|}{u^{\frac{1}{2}}}du$.(Hint: Show $(\int_0^\infty(\int_0^\infty |k(xy)f(y)|dy)^{\frac{1}{2}}dx)^{\frac{1}{2}} \leq \int_0^\infty \frac{|k(u)|}{u^{\frac{1}{2}}}dudx\|f\|_2$ by changing variable $u = xy$, applying the Minkowski's inequality for integrals, and changing variable $z = u/x$.)

Proof. By the proof of 6.21 with $p = 2$, we have for $f \in L^2(0,\infty)$ and $x \in (0,\infty)$, due to $f \in L^2(0,\infty)$

$$\left|T(f)(x)\right| \leq \int_0^\infty \left|k(x,y)f(y)\right|dy \leq \left(\int_0^\infty \frac{|k(u)|}{u^{\frac{1}{2}}}du\right)\left(\int_0^\infty f(x)^2dx\right)^{\frac{1}{2}} = \left(\int_0^\infty \frac{|k(u)|}{u^{\frac{1}{2}}}du\right)\|f\|_2 < \infty$$

This immediately implies that $T(f)(x) \in \mathbb{R}$. There is no doubt $T(f)$ is a well-define measure function. We prove that $\|T(f)\|_2 < \infty$. We have for $f \in L^2(0,\infty)$ due to the proof in the previous question with $g(x) \equiv 1$, the given condition and Definition 6.1.1.(b)

$$\|T(f)\|_2 = \left(\int_0^\infty \left|T(f)(x)\right|^2 dx\right)^{\frac{1}{2}} = \left(\int_0^\infty \left(\int_0^\infty k(xy)f(y)dy\right)^2 dx\right)^{\frac{1}{2}} \leq \int_0^\infty \frac{|k(u)|}{u^{\frac{1}{2}}}du\|f\|_2 < \infty$$

But such f was chosen randomly. By Exercise 5.3, we have $\|T\| \leq \int_0^\infty \frac{|k(u)|}{u^{\frac{1}{2}}}du$. It remains to prove the linearity of T. For any $\alpha \in \mathbb{R}$, $f_1, f_2 \in L^2(0,\infty)$ and $x \in (0,\infty)$, we have

$$T(\alpha f_1 + f_2)(x) = \int_0^\infty k(xy)(\alpha f_1 + f_2)(y)dy = \alpha\int_0^\infty k(xy)f_1(y)dy + \int_0^\infty k(xy)f_2(y)dy = (\alpha T(f_1) + T(f_2))(x).$$

$\square$

Homework 13 — Math 5323

Due: Fridays, 3 May 2022, by 11:59 p.m. CDT

Question 6.22: (A generalized Holder inequality) Let $1 \leq p_j \leq \infty$ and $\sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{r} \leq 1$. Let $\{f_j\}$ be measurable functions. Prove that,

$$\Big\| \prod_{j=1}^{n} f_j \Big\|_r \leq \prod_{j=1}^{n} \|f_j\|_{p_j} \tag{1}$$

Proof. We want to prove this by induction on $n \in \Big\{2, 3, \cdots\Big\}$. For the base step $n = 2$, we are required to prove that

$$\Big\| \prod_{j=1}^{2} f_j \Big\|_r \leq \prod_{j=1}^{2} \|f_j\|_{p_j}$$

which is equivalent to

$$\|f_1 f_2\|_r \leq \|f_1\|_{p_1} \|f_2\|_{p_2}. \tag{2}$$

We have from (4),

$$\int_X \Big|fg\Big|^{\frac{pq}{p+q}} \leq \Big(\int_X \Big|f\Big|^q\Big)^{\frac{1}{p}} \Big(\int_X \Big|g\Big|^p\Big)^{\frac{1}{q}}$$

where we denote $q := p_2$,$p := p_1$ $f := f_1$ and $g := f_2$, which implies that after taking both side of the square $\frac{1}{r}$ root, we have

$$\Big(\int_X \Big|fg\Big|^r\Big)^{\frac{1}{r}} \leq \Big(\int_X \Big|f\Big|^q\Big)^{\frac{1}{q}} \Big(\int_X \Big|g\Big|^p\Big)^{\frac{1}{p}}.$$

Finally, by Definition 6.1.1(a), we proved (2). Now we want to finish the inductive step. We assume (1) holds for $n \geq 2$. We are required to prove it holds for $n + 1$. But actually we have

$$\begin{aligned} \Big\| f_1 \cdots f_n f_{n+1} \Big\|_r &\leq \Big\| f_1 \cdots f_n \Big\|_{r_{n+1}} \Big\| f_{n+1} \Big\|_{p_{n+1}} && (3) \\ &\leq \Big(\prod_{j=1}^{n} \|f_j\|_{p_j} \Big) \Big\| f_{n+1} \Big\|_{p_{n+1}} && (4) \\ &= \prod_{j=1}^{n+1} \|f_j\|_{p_j} && (5) \end{aligned}$$

where (3) is due to the base step where we denote $\frac{1}{r_{n+1}} = \frac{1}{p_1} + \cdots + \frac{1}{p_n}$ and $f_1 \cdots f_n$ are measurable, and (4) is due to the inductive hypothesis.

$\square$

Question 6.23: Prove that if f and g are Lebesgue measurable over $\mathbb{R}^n$, then $h(x, y) := f(x-y)g(y)$ is Lebesgue measurable from $\mathbb{R}^{2n}$ to $\mathbb{R}$. (Hint: $f(x-y)g(y)$ is a composition of $f(ug(v))$ with the linear transformation $u = x - y$, $v = y$.)

Proof. We finish the proof in 3 steps.

Step 1: We define $F : \mathbb{R}^{2n} \to \mathbb{R}$ defined by $F(u, v) := f(u)$ and $G : \mathbb{R}^{2n} \to \mathbb{R}$ defined by $G(u, v) := g(v)$. We choose $\alpha \in \mathbb{R}$ randomly. Then $F^{-1}((\alpha, \infty)) = f^{-1}((\alpha, \infty)) \times \mathbb{R}^n$ is measurable by Definition and Proposition 2.1.4(a) and Definition 2.6.1 where f is measurable and $\mathbb{R}^n \in \mathcal{B}_{\mathbb{R}^n}$. But such $\alpha \in \mathbb{R}$ was chosen randomly. We proved that F is measurable. Similarly we can prove that G is measurable.

Step 2: We define $\alpha : \mathbb{R}^{2n} \to \mathbb{R}$ by $\alpha(u, v) := f(u)g(v)$. Then we have $\alpha = FG$ and we have α is measurable due to Proposition 2.1.5.

Step 3: We denote $T(x, y) := \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ for $x, y \in \mathbb{R}^n$ and we consider T is a function from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ We denote

$$A := \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix}$$

Then we have by Definition of α and Definition of h, for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{R}^{2n}$

$$\alpha(T(x, y)) = \alpha(x - y, y) = f(x - y)g(y) = h(x, y)$$

Since A is singular where $\det(A) = 1 \neq 0$, by Theorem 2.7.5, we have that $h = \alpha \circ T$ is measurable.

$\square$

Question 6.24: Prove that if f is continuous and it is integrable over bounded subset of $\mathbb{R}^n$ and $g \in C^k(\mathbb{R}^n)$ has compact support, then $f * g \in C^k(\mathbb{R}^n)$.

Proof. We denote $A := supp(g)$ for some compact set $A \subset \mathbb{R}^n$ and we know A is bounded due to Heine-Borel Theorem. We define $f_1 := f\chi_A$. We denote $B \subset \mathbb{R}^n$ for some bounded subset B such that $f \in L^1(B)$. We denote $C := A \cap B$ for some $C \subset \mathbb{R}^n$. Now we have

$$\int_C \Big|f_1\Big| d\mu = \int_{A\cap B} \Big|f\chi_A\Big| d\mu = \int_{A\cap B} \Big|f\Big| d\mu \leq \int_{A\cap B} \sup_{A\cap B}(|f|) d\mu = M\mu(A \cap B) < \infty$$

where the first equality is due to Definition of C, and last is due to the extreme value theorem since f is continuous where we denote $M := \sup_C(|f|)$ for some $M \in \mathbb{R}$ and we know $\mu(A \cap B) < \infty$ since A and B are both bounded. By Definition 2.3.1. we proved that $f_1 \in L^1(C)$. By Definition 6.4.1 and Definition of support domains, we have

$$\begin{aligned}
f_1 * g(x) &= \int_{\mathbb{R}^n} f_1(y)g(x-y)dy \\
&= \int_{\mathbb{R}^n} f\chi_A(y)g(x-y)dy \\
&= \int_A f\chi_A(y)g(x-y)dy + \int_{\mathbb{R}^n\setminus A} f\chi_A(y)g(x-y)dy \\
&= \int_A f(y)g(x-y) + \int_{\mathbb{R}^n\setminus A} f(y)g(x-y)dy \\
&= \int_{\mathbb{R}^n} f(y)g(x-y)dy \\
&= f * g(y),
\end{aligned}$$

which says that it is enough to prove that $f_1 * g \in C^k(\mathbb{R}^n)$. Since $f_1 \in L^1$ and $g \in C^k(\mathbb{R}^n)$, by Proposition 6.4.6, it is enough to prove that $\partial^\alpha g$ is bounded for $|\alpha| \leq k$. There is no doubt that $\partial^\alpha g$ is continuous for $|\alpha| < k$ by Definition of $C^k(\mathbb{R}^n)$. Then by the extreme value theorem and Definition of compact supports, we have $\sup_{x\in\mathbb{R}^n} |(\partial^\alpha g)(x)| = \sup_{x\in A} |(\partial^\alpha g)(x)| < \infty$. So we are done. $\square$

Question 6.25: Let $1 \le p \le \infty$ and q be conjugate to p. Prove that for any $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, $f * g$ exists, $f * g \in L^\infty(\mathbb{R})$, and

$$\|f * g\|_\infty \le \|f\|_p \|g\|_q.$$

Proof. We choose $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ randomly. By Proposition 6.4.4, we have that $f * g(x)$ exists for every $x \in \mathbb{R}$ and $h := f * g$ is bounded and uniformly continuous. By Definition 6.1.10(a), to prove that $\|h\|_\infty < \infty$, it is enough to find $0 \le \alpha < \infty$ such that $\mu(\left\{x \in \mathbb{R} : |h(x)| > \alpha\right\}) = 0$. We denote $M := \sup_{x \in \mathbb{R}} |h(x)|$ for some $M \in \mathbb{R}$ by Definition of boundedness. Then by Definition of sup, we have

$$A := \left\{x \in \mathbb{R} : |h(x)| > M + 1\right\} = \varnothing$$

which immediately implies that $\mu(A) = 0$ by Definition 1.3.1. We prove the more general case which is called Young's inequality for convolutions and the assertion is : Suppose $1 \le p, q, r \le \infty$ and $f \in L^P(\mathbb{R})$ and $g \in L^q(\mathbb{R})$. Prove that

$$\|f * g\|_r \le \|f\|_p \|g\|_q \text{ where } \tfrac{1}{r} + 1 = \tfrac{1}{p} + \tfrac{1}{q}$$

□

We denote $a := \frac{p}{r}$ and $b := \frac{q}{r}$ and we find that

$$\left|\int f(y)g(x-y)dy\right| \le \|f(y)g(x-y)\|_{L^1(y)} \tag{6}$$

$$= \|(f(y)^{\frac{p}{r}} g(x-y)^{\frac{q}{r}}) f(y)^{1-\frac{p}{r}} g(x-y)^{1-\frac{q}{r}}\|_{L^1(y)} \tag{7}$$

Since we have

$$\frac{1}{r} + \frac{1}{pr/(r-p)} + \frac{1}{qr/(r-q)} = \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1,$$

we apply Generalized Holder's inequality to (7) to have

$$(7) \le \|f(y)^{\frac{p}{r}} g(x-y)^{\frac{q}{r}}\|_{L^r(y)} \|f(y)^{1-\frac{p}{r}}\|_{L^{\frac{r-p}{pr}}} \|g(x-y)^{1-\frac{q}{r}}\|_{r^{\frac{r-q}{qr}}} = \left(\int |f(x)|^p |g(x-y)|^q dy\right)^{\frac{1}{r}} \|f\|_{L^p}^{\frac{r-p}{r}} \|g\|_{L^q}^{\frac{r-q}{r}}$$

Finally, we compute that

$$\begin{aligned}
\|f * g\|_{L^r}^r &\le \left(\int\int |f(x)|^p |g(x-y)|^q dy dx\right) \|f\|_{L^p}^{r-p} \|g\|_{L^q}^{r-q} \\
&= \left(\|g\|_{L^q}^q \int |f(x)|^p dx\right) \|f\|_{L^p}^{r-p} \|g\|_{L^q}^{r-q} \\
&= \|f\|_{L^p}^p \|g\|_{L^q}^q \|f\|_{L^p}^{r-p} \|g\|_{L^q}^{r-q} \\
&= \|f\|_{L^p}^r \|g\|_{L^q}^r
\end{aligned}$$

Taking rth roots completes the computation.

Question 6.26: Prove that if $f \in L^1$ and $g \in L^p$ for $1 \leq p \leq \infty$, then $f * g(x)$ exists for a.e. x, $f * g \in L^p$ and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p$$

(Hint: Theorem 6.3.2.)

Proof. We consider $k(x,y) = f(x-y)$ in the Theorem 6.3.2. Since $f \in L^1$, by Definition 6.1.1(a), we have

$$\int |k(x,y)| d\mu(x) = \int |f(x-y)| d\mu(x) = \int |f(z)| dz < \infty \text{ for a.e. } y \in \mathbb{R} \text{ and for a.e. } x \in \mathbb{R}.$$

Then plugging $k(x,y) = f(x,y)$ in Theorem 6.3.2, since $g \in L^p(\mathbb{R})$, we have $f(x-y)g(y) \in L^1_y$. Then by Definition of convolution and Definition 6.1.1(a), we have for any $x \in \mathbb{R}$

$$\Big| f * g(x) \Big| = \left| \int_{\mathbb{R}} f(x-y) g(y) d\nu(y) \right| \leq \int_{\mathbb{R}} \Big| f(x-y) g(y) \Big| d\nu(y) < \infty$$

which immediately implies that $f * g(x)$ exists for a.e. x. Also we have by Definition of convolution

$$\int_{\mathbb{R}} \Big| (f * g)(x) \Big|^p d\mu(x) = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y) g(y) d\nu(y) \right|^p d\mu(x) = \int_{\mathbb{R}} \Big| T(g)(x) \Big|^p d\mu(x) < \infty$$

which immediately implies that by Definition 6.1.1(b) $f * g \in L^p(\mathbb{R})$. By Definition of convolution, we have

$$\begin{aligned} \|f * g\|_p &= \left\| \int_{\mathbb{R}} g(x-y) f(y) d\nu(y) \right\|_{p,\mu(x)} && (8) \\ &\leq \int_{\mathbb{R}} \|g(x-y) f(y)\|_{p,\mu(x)} d\nu(y) && (9) \end{aligned}$$

due to Theorem 6.3.3. where $f(x-y)g(y)$ is $\nu(y)$-integrable function. Now from (9), we have

$$\begin{aligned} & \int_{\mathbb{R}} \|g(x-y) f(y)\|_{p,\mu(x)} d\nu(y) && (10) \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |g(x-y) f(y)|^p d\mu(x) \right)^{\frac{1}{p}} d\nu(y) && (11) \\ &= \int_{\mathbb{R}} \left(|f(y)| \left(\int_{\mathbb{R}} |g(x-y)|^p d\mu(x) \right)^{\frac{1}{p}} \right) d\nu(y) && (12) \\ &= \int_{\mathbb{R}} |f(y)| \left(\int_{\mathbb{R}} |g(u)|^p d\mu(u) \right)^{\frac{1}{p}} d\nu(y) && (13) \\ &= \left(\int_{\mathbb{R}} |f(y)| d\nu(y) \right) \left(\int_{\mathbb{R}} |g(x)|^p d\mu(x) \right)^{\frac{1}{p}} && (14) \\ &= \|f\|_1 \|g\|_p && (15) \end{aligned}$$

where (11) and (15) is due to Definition 6.1.1(a). Combing (9) and (15), we proved that

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

$\square$