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Group dualities, T-dualities, and twisted K-theory. (English summary)
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The paper under review builds upon a solid motivation arising from mathematics and physics: the intriguing occurrence of T-duality in the context of Langlands duality of compact simple Lie groups [C. Daenzer and E. van Erp, Adv. Theor. Math. Phys. **18** (2014), no. 6, 1267–1285; MR3285609; U. Bunke and T. Nikolaus, Rev. Math. Phys. **27** (2015), no. 5, 1550013; MR3361543] as well as the twisted K-theory of compact Lie groups initiated by the study of Wess-Zumin-Witten (WZW) models [V. Braun, J. High Energy Phys. **2004**, no. 3, 029; MR2061550; C. L. Douglas, Topology **45** (2006), no. 6, 955–988; MR2263220]. The authors make a number of important clarifications about this interesting connection between the two kinds of dualities, and they illustrate an alternative, demonstrably effective way of computing twisted K-theory of compact Lie groups which is applicable to more general cases and help to illuminate the results by Braun and Douglas. These are carried out with various carefully chosen computational examples and an impressive amount of results, details and techniques organized in a painstaking way.

Recall that T-dualization is a transformation which takes a principal torus bundle $p\colon E\to X$ with a background B-field h (which can be thought of as an element in $H^3(E,\mathbb{Z})$) to its principal dual torus bundle $p^\vee\colon E^\vee\to X$ with dual B-field $h^\vee\in H^3(E^\vee,\mathbb{Z})$ satisfying some conditions relating the Chern classes of the bundles and the pushforward of h and $h^\vee$ through projections. Two compact, connected and simple Lie groups G and $G^\vee$ are said to be Langlands dual to each other if the embedding of the root lattice $\Phi(G,T)$ of G into its weight lattice $\Lambda^*(G)$ is isomorphic to that of the coroot lattice $\Phi^\vee(G^\vee,T^\vee)$ of $G^\vee$ into its coweight lattice $\Lambda(G^\vee)$. Note that the maximal tori T and $T^\vee$ are dual tori, and when G is not of type B_n or C_n when $n\geq 3$, the flag manifolds G/T and $G^\vee/T^\vee$ are isomorphic. Thus G and $G^\vee$ in these cases can be viewed as dual principal torus bundles over $X:=G/T$. The main result of Daenzer and van Erp and of Bunke and Nikolaus is that there exist $h\in H^3(G,\mathbb{Z})$ and $h^\vee\in H^3(G^\vee,\mathbb{Z})$ such that (G,h) and $(G^\vee,h^\vee)$ are T-dual to each other. However, they did not point out explicitly what h and $h^\vee$ are and how they are related to the generators of $H^3(G,\mathbb{Z})$ and $H^3(G^\vee,\mathbb{Z})$.

In Section 2 of the paper under review, the authors address the aforementioned gap. They begin by answering the following seemingly innocuous question, which is of independent interest. Consider the universal cover $\widetilde{G}$ and the map $\pi^*\colon H^3(G,\mathbb{Z})\to H^3(\widetilde{G},\mathbb{Z})$ induced by projection. In most cases, both $H^3(G,\mathbb{Z})$ and $H^3(\widetilde{G},\mathbb{Z})$ are isomorphic to $\mathbb{Z}$. Then what is $\pi^*(1)$? They show in Theorem 1 that $\pi^*(1)$ is either 1 or 2 and give a case-by-case analysis which turns out to be quite elaborate, involving studying the kernel of π^* with the coefficient replaced by the finite field $\mathbb{Z}_q$ and invoking results on $H^*(G,\mathbb{Z}_q)$ [A. Borel, Ann. of Math. (2) **57** (1953), 115–207; MR0051508]. We note that, while being essential in the authors' explicit description of h and $h^\vee$, Theorem 1 is also useful when applied in combination with the Freed-Hopkins-Teleman Theorem [D. S. Freed, M. J. Hopkins and C. Teleman, Ann. of Math. (2) **174** (2011), no. 2, 947–1007; MR2831111] in understanding the map $\pi^*\colon K^\tau_G(G)\to K^{\pi^*\tau}_{\widetilde{G}}(\widetilde{G})$ and thus yielding the integrable weights of LG in relation to those of $L\widetilde{G}$ which are well known.

After carefully working out several examples of pairs of low-rank compact Lie groups which are both Langlands dual and T-dual, the authors go on to completely settle the nature of the T-dualities coming from Langlands duality with Theorem 11: if G is a compact simply connected simple Lie group not of type B_n or C_n for $n \geq 3$, $h^\vee$ is the generator of $H^3(G^\vee, \mathbb{Z})$, and $h = \pi^*(h^\vee)$ as specified by Theorem 1, then (G, h) and $(G^\vee, h^\vee)$ are T-dual. Moreover, inspired by the isomorphism of twisted K-theories of T-dual pairs [P. G. Bouwknegt, J. Evslin and V. Mathai, Comm. Math. Phys. **249** (2004), no. 2, 383–415; MR2080959], they also look into $K^*(G, h)$ and $K^*(G^\vee, h^\vee)$ and find that these groups all vanish except in the case $G = SU(2)$ and $G^\vee = SO(3)$ when both groups are $\mathbb{Z}_2$.

The latter part of Section 2 is devoted to an alternative, easier method of computing $K^*(G, h)$ for general connected compact simple Lie groups and h, representing an improvement of the methods employed by Braun and Douglas, which are applicable only to the simply connected case. This comes in the form of a twisted version of Segal spectral sequence for complex K-theory of fiber bundles. They apply this method to $G = SU(n+1)$, $Sp(n)$, G_2, $PSU(n+1)$ and $PSp(n)$, which can be realized as fiber bundles, and h satisfying certain primality conditions. They also note that the 'somewhat puzzling results' of Braun and Douglas can be better explained by this spectral sequence approach. For example, the 'strange denominator' appearing in the formula for the exponent of the group $K^*(SU(n+1), h)$,

$$\frac{h}{\gcd(h, \operatorname{lcm}(1, 2, \cdots, n))},$$

in fact is related, at least in the case when h is relatively prime to $n!$, to the collapsing of the spectral sequence applied to the fibration

$$SU(n) \to SU(n+1) \to S^{2n+1},$$

which in turn is due to the triviality of the fibration after inverting $n!$, which is the order of $\pi_{2n}(SU(n))$ asserted by R. H. Bott's theorem [Michigan Math. J. **5** (1958), 35–61; MR0102803].

The paper ends with a formulation of T-duality of orientifold string theories on connected semi-simple complex Lie groups, and a natural isomorphism of KR-theory of such a Lie group equipped with holomorphic and anti-holomorphic involutions (Theorem 21), as predicted by the T-duality of orientifold string theories. *Chi-Kwong Fok*

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