Journal of Mathematical Analysis
ISSN: 2217-3412, URL: www.ilirias.com/jma
Volume 15 Issue 5 (2024), Pages 55-68
https://doi.org/10.54379/jma-2024-5-4

CHARACTERIZATIONS OF LINEARLY INDEPENDENT FUNCTIONS

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Abstract. For functions $f_1, \ldots, f_n$ on a set D, we characterize their linear independence with an invertible matrix from their values at n distinct points in D. With the matrix, the pointwise convergence of a sequence $\{g_k\}$ of functions in the $span\{f_1, \cdots, f_n\}$ is shown to be equivalent to those of the sequences of the coordinates of g_ks in the span. When f_is are bounded, a pointwise convergent sequence $\{g_k\}$ must uniformly converge to a function in the span. It turns out that the limit of a convergent sequence $\{g_k\}$ inherits the continuity, differentiability, and integrability of f_is. Furthermore the (pointwise or uniform) convergence of a sequence of solutions of an n-th order constant coefficients linear differential equation is completely determined by that of the sequence of relevant initial conditions.

1. Introduction

Consider a set of functions $f_1, f_2, \ldots, f_n$ from a nonempty set D to $\mathbb{K}$ (the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$). The set is said to be linearly dependent if there exist $c_1, c_2, \ldots, c_n$ in $\mathbb{K}$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all } x \in D. \tag{1.1}$$

The set of functions is said to be linearly independent on D provided that it is not linearly dependent.

For $(n-1)$ times differentiable functions $f_1, f_2, \ldots, f_n$ on an interval I, it is known that their linear dependence implies that their Wronskian vanishes in I. Hence, if the Wronskian of these functions is not zero at some point in I, then they must be linearly independent. However, linearly independent functions may have vanishing Wronskian. For example, the Wronskian of functions $f_1(x) = x^2$ and $f_2(x) = x|x|$ vanishes on $\mathbb{R}$ [10] but they are linearly independent. So, without extra conditions, a vanishing Wronskian of functions does not completely characterize their linear dependence. Many relevant results on linear dependence of functions have been obtained by studying their Wronskian, see [1]-[11] and [13]-[14]. However, these results are not applicable to the cases where functions are not differentiable.

1991 *Mathematics Subject Classification.* 40A30, 15A03, 34A12.
Key words and phrases. Linear independence, uniform convergence, initial value problem.

Submitted on December 22, 2023. Accepted August 6, 2024. Published September 20, 2024.
Communicated by Pablo M. Berna.

We note that the linear independence of $f_1, f_2, \ldots, f_n$ implies that for any $C = (c_1, c_2, \ldots, c_n)^t \neq 0$ there exists at least one point $x \in D$ such that (1.1) fails. Are there more such points in D when $n > 1$? Can we characterize the linear independence of these functions in terms of their values at such points? For the function space spanned by these linearly independent functions, does every pointwise convergent sequence in it uniformly converge? Does the limit function of a pointwise convergent sequence in the spanned space inherit the continuity, differentiability, and integrability of the sequence? Recall that there are n linearly independent solutions for an n-th order constant coefficients linear differential equation. Are there any relations between a sequence of initial conditions and the sequence of solutions of relevant initial value problems? This paper aims to present positive answers to these questions.

2. Linear Independence of Functions

As we see, for linearly independent functions $f_1, f_2, \ldots, f_n$ on D and for any $0 \neq C \in \mathbb{K}^n$ there exists at least one point x in D such that (1.1) fails. Our first result in this section states that for such functions there must exist n distinct points in D such that (1.1) fails.

Theorem 2.1. Let $f_1, f_2, \ldots, f_n$ be functions defined on a nonempty set D. Then $f_1, f_2, \ldots, f_n$ are linearly independent on D if and only if there exist distinct points $x_1, x_2, \ldots, x_n$ in D such that

$$[f_i(x_j)] := \begin{bmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(x_1) & f_n(x_2) & \cdots & f_n(x_n) \end{bmatrix}$$

is invertible. Hence $f_1, f_2, \ldots, f_n$ are linearly dependent if and only if for any distinct points $x_1, x_2, \ldots, x_n$ in D the matrix $[f_i(x_j)]$ is not invertible.

Proof. Denote

$$F_i := [f_i(x_1)\ f_i(x_2)\ \cdots\ f_i(x_n)]^t \in \mathbb{K}^n \quad \text{for } i = 1, \ldots, n.$$

Then $[f_i(x_j)]$ is invertible iff $F_1, F_2, \ldots, F_n$ are linearly independent. So it suffices to show that $f_1, f_2, \ldots, f_n$ are linearly independent on D iff there exist distinct points $x_1, x_2, \ldots, x_n$ in D such that $F_1, F_2, \ldots, F_n$ are linearly independent.

Firstly we prove the sufficiency. Suppose that there exist $x_1, x_2, \ldots, x_n$ in D such that $F_1, F_2, \ldots, F_n$ are linearly independent. Then $f_1, f_2, \ldots, f_n$ must be linearly independent. Otherwise, suppose that they are linearly dependent. Then there exists $(c_1, c_2, \ldots, c_n) \neq (0, 0, \ldots, 0)$ such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all } x \in D.$$

Taking $x = x_1, x_2, \ldots, x_n$, we get $c_1 F_1 + c_2 F_2 + \cdots + c_n F_n = 0$, that is, $F_1, \ldots, F_n$ are linearly dependent, contradicting the assumption.

To prove the necessity, let $f_1, f_2, \ldots, f_n$ be linearly independent on D. Then for each $i = 1, \ldots, n$ there exists $x_i \in D$ such that $f_i(x_i) \neq 0$.

For $n = 2$, it is easy to see that either $f_1(x) f_2(x) = 0$ for all $x \in D$ or there exists $x \in D$ such that $f_1(x) f_2(x) \neq 0$.

If $f_1(x)f_2(x) = 0$ for all $x \in D$, then for each $x \in D$ either $f_1(x) = 0$ or $f_2(x) = 0$. Since there exist x_1, x_2 in D such that $f_1(x_1) \neq 0$ and $f_2(x_2) \neq 0$, $f_2(x_1) = 0$ and $f_1(x_2) = 0$. It follows that x_1 and x_2 are distinct and

$$\begin{bmatrix} f_1(x_1) \\ f_1(x_2) \end{bmatrix} = \begin{bmatrix} f_1(x_1) \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} f_2(x_1) \\ f_2(x_2) \end{bmatrix} = \begin{bmatrix} 0 \\ f_2(x_2) \end{bmatrix}$$

are linearly independent.

If there exists $x \in D$ such that $f_1(x)f_2(x) \neq 0$, then there must exist distinct points x_1, x_2 in D such that

$$\begin{bmatrix} f_1(x_1) \\ f_1(x_2) \end{bmatrix} \text{ and } \begin{bmatrix} f_2(x_1) \\ f_2(x_2) \end{bmatrix}$$

are linearly independent. Otherwise, for any distinct points x_1, x_2 in D there exists $(c_1, c_2) \neq (0, 0)$ such that

$$c_1 \begin{bmatrix} f_1(x_1) \\ f_1(x_2) \end{bmatrix} + c_2 \begin{bmatrix} f_2(x_1) \\ f_2(x_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Note that, for a point x_1 satisfying $f_1(x_1)f_2(x_1) \neq 0$, we have $c_1 \neq 0$ and $c_2 \neq 0$. It follows from $c_1 f_1(x_1) + c_2 f_2(x_1) = 0$ that

$$f_1(x_1) = c f_2(x_1) \text{ with } c := -\frac{c_2}{c_1}.$$

In addition, for any $x \in D$ with $x \neq x_1$, there exists $(d_1, d_2) \neq (0, 0)$ such that

$$d_1 \begin{bmatrix} f_1(x_1) \\ f_1(x) \end{bmatrix} + d_2 \begin{bmatrix} f_2(x_1) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This with $f_1(x_1) = c f_2(x_1)$ implies that $(d_1 c + d_2) f_2(x_1) = 0$. Hence

$$d_2 = -d_1 c \text{ and } d_1[f_1(x) - c f_2(x)] = d_1 f_1(x) - d_1 c f_2(x) = d_1 f_1(x) + d_2 f_2(x) = 0.$$

Since $f_1(x_1)f_2(x_1) \neq 0$ and $(d_1, d_2) \neq (0, 0)$, $d_1 \neq 0$. Thus $f_1(x) = c f_2(x)$ for all $x \in D$, a contradiction.

For $n = k \geq 2$, suppose that there exist distinct points $x_1, x_2, \ldots, x_k$ in D such that for any $(c_1, c_2, \ldots, c_k) \neq (0, 0, \ldots, 0)$ there holds

$$c_1 \begin{bmatrix} f_1(x_1) \\ f_1(x_2) \\ \vdots \\ f_1(x_k) \end{bmatrix} + c_2 \begin{bmatrix} f_2(x_1) \\ f_2(x_2) \\ \vdots \\ f_2(x_k) \end{bmatrix} + \cdots + c_k \begin{bmatrix} f_k(x_1) \\ f_k(x_2) \\ \vdots \\ f_k(x_k) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.1}$$

Then, for $n = k + 1$, to show the conclusion desired to be valid, we suppose that for any $y_1, y_2, \ldots, y_{k+1}$ in D there exists $(d_1, d_2, \ldots, d_{k+1}) \neq (0, 0, \ldots, 0)$ such that

$$d_1 \begin{bmatrix} f_1(y_1) \\ f_1(y_2) \\ \vdots \\ f_1(y_{k+1}) \end{bmatrix} + d_2 \begin{bmatrix} f_2(y_1) \\ f_2(y_2) \\ \vdots \\ f_2(y_{k+1}) \end{bmatrix} + \cdots + d_{k+1} \begin{bmatrix} f_{k+1}(y_1) \\ f_{k+1}(y_2) \\ \vdots \\ f_{k+1}(y_{k+1}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.2}$$

In particular, for the points $x_1, x_2, \ldots, x_k$ in (2.1) and any fixed x_{k+1} in $D \setminus \{x_1, x_2, \ldots, x_k\}$ (which is nonempty due to the linear independence of $f_1, f_2, \ldots, f_{k+1}$), there exists $(\overline{d}_1, \overline{d}_2, \dots, \overline{d}_{k+1}) \neq (0, 0, \dots, 0)$ such that

$$\overline{d}_1 \begin{bmatrix} f_1(x_1) \\ f_1(x_2) \\ \vdots \\ f_1(x_{k+1}) \end{bmatrix} + \overline{d}_2 \begin{bmatrix} f_2(x_1) \\ f_2(x_2) \\ \vdots \\ f_2(x_{k+1}) \end{bmatrix} + \cdots + \overline{d}_{k+1} \begin{bmatrix} f_{k+1}(x_1) \\ f_{k+1}(x_2) \\ \vdots \\ f_{k+1}(x_{k+1}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.3}$$

From (2.3) and (2.1), we see that $\overline{d}_{k+1} \neq 0$. If we denote $M := [f_{ij}]_{k \times k}$ with $f_{ij} = f_j(x_i)$ for $i = 1, \dots, k$ and $\overline{d} := [\overline{d}_1 \ \overline{d}_2 \ \cdots \ \overline{d}_k]^t$, then M is invertible and it follows from (2.3) that

$$\begin{bmatrix} f_{k+1}(x_1) \\ f_{k+1}(x_2) \\ \vdots \\ f_{k+1}(x_k) \end{bmatrix} = -\overline{d}_{k+1}^{-1} M \overline{d}. \tag{2.4}$$

Now for any $x \in D \setminus \{x_1, x_2, \dots, x_k\}$, by (2.2), there exist $d \in \mathbb{K}^k$ and $d_{k+1} \in \mathbb{K}$ such that $(d_1, d_2, \dots, d_{k+1}) \neq (0, 0, \dots, 0)$,

$$M(d - d_{k+1}\overline{d}_{k+1}^{-1}\overline{d}) = Md - d_{k+1}\overline{d}_{k+1}^{-1} M\overline{d} = 0, \text{ and } \sum_{j=1}^{k} d_j f_j(x) + d_{k+1} f_{k+1}(x) = 0.$$

Obviously $d_{k+1} \neq 0$. In addition, since M is invertible, $d - d_{k+1}\overline{d}_{k+1}^{-1}\overline{d} = 0$. Thus $d_j = d_{k+1}\overline{d}_{k+1}^{-1}\overline{d}_j$ for $j = 1, \dots, k$. So

$$f_{k+1}(x) = -d_{k+1}^{-1} \sum_{j=1}^{k} d_j f_j(x) = -\overline{d}_{k+1}^{-1} \sum_{j=1}^{k} \overline{d}_j f_j(x).$$

This with (2.4) shows that $f_1, \dots, f_{k+1}$ are linearly dependent, a contradiction. And hence the conclusion desired for $n = k + 1$ is valid. Therefore, by induction, the conclusion is valid for all $n \in \mathbb{N}$. □

For functions $f_1, f_2, \dots, f_n$ on D, there exist an invertible matrix $[c_{ij}]$ and functions $e_1, e_2, \dots, e_n$ on D such that

$$f_i(x) = c_{i1}e_1(x) + c_{i2}e_2(x) + \cdots + c_{in}e_n(x) \quad \text{for } 1 \leq i \leq n \text{ and } x \in D.$$

If $e_1, e_2, \dots, e_n$ are linearly independent on D, then, by Theorem 2.1, it is easy to see that $f_1, f_2, \dots, f_n$ are linearly independent on D. Next result states that the linear independence of $f_1, f_2, \dots, f_n$ must conversely imply that of $e_1, e_2, \dots, e_n$.

Theorem 2.2. Let $f_1, f_2, \dots, f_n$ be linearly independent functions on D. If there exist a matrix $[c_{ij}]$ and functions $e_1, e_2, \dots, e_n$ on D such that

$$f_i(x) = c_{i1}e_1(x) + c_{i2}e_2(x) + \cdots + c_{in}e_n(x) \quad \text{for } 1 \leq i \leq n \text{ and } x \in D,$$

then $[c_{ij}]$ is invertible and $e_1, e_2, \dots, e_n$ are linearly independent on D. In addition, for $[d_{ij}] = [c_{ij}]^{-1}$ there holds

$$e_i(x) = d_{i1}f_1(x) + d_{i2}f_2(x) + \cdots + d_{in}f_n(x) \quad \text{for } x \in D.$$

Proof. Since $f_1, f_2, \ldots, f_n$ are linearly independent on D, by Theorem 2.1, there exist n distinct points $x_1, x_2, \ldots, x_n$ in D such that $[f_i(x_j)]$ is invertible. By assumption, we have

$$\begin{bmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ f_2(x_1) & \cdots & f_2(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} e_1(x_1) & \cdots & e_1(x_n) \\ e_2(x_1) & \cdots & e_2(x_n) \\ \vdots & \ddots & \vdots \\ e_n(x_1) & \cdots & e_n(x_n) \end{bmatrix}.$$

This shows that both $[c_{ij}]$ and $[e_i(x_j)]$ are invertible and hence, by Theorem 2.1, $e_1, e_2, \ldots, e_n$ are linearly independent on D. So, there exists $[d_{ij}]$ such that

$$e_i(x) = d_{i1}f_1(x) + d_{i2}f_2(x) + \cdots + d_{in}f_n(x) \quad \text{for } 1 \le i \le n \text{ and } x \in D.$$

Taking $x = x_1, x_2, \ldots, x_n$, we obtain

$$\begin{bmatrix} e_1(x_1) & \cdots & e_1(x_n) \\ e_2(x_1) & \cdots & e_2(x_n) \\ \vdots & \ddots & \vdots \\ e_n(x_1) & \cdots & e_n(x_n) \end{bmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ d_{21} & \cdots & d_{2n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{bmatrix} \begin{bmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ f_2(x_1) & \cdots & f_2(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{bmatrix}.$$

Thus

$$[e_i(x_j)] = [d_{ik}][f_k(x_j)] = [d_{ik}][c_{km}][e_m(x_j)] = [d_{mk}][c_{ki}][e_i(x_j)]$$

and hence $[d_{ik}][c_{kj}] = I_{n\times n}$, that is, $[d_{ij}] = [c_{ij}]^{-1}$. □

Given functions $f_1, \ldots, f_n : D \to \mathbb{K}$, we denote

$$span\{f_1, \ldots, f_n\} := \{f : f(x) = c_1f_1(x) + \cdots + c_nf_n(x), x \in D, c_i \in \mathbb{K}\ (1 \le i \le n)\}$$

associated with the norm $\|f\| := \sup\{|f(x)| : x \in D\}$ for $f \in span\{f_1, \ldots, f_n\}$. As another application of Theorem 2.1, next result about interpolation can be obtained by the linear independence of functions.

Corollary 2.3. Let f be a function from D to $\mathbb{K}$. If there exist distinct points $x_1, x_2, \ldots, x_n$ in D and functions $f_1, f_2, \ldots, f_n$ on D such that $[f_i(x_j)]_{n\times n}$ is invertible, then there exists a unique function $g \in span\{f_1, f_2, \ldots, f_n\}$ such that

$$g(x_j) = f(x_j) \quad \text{for } j = 1, 2, \ldots, n.$$

Proof. Taking $C^t \in \mathbb{K}^n$ such that $C[f_i(x_j)] = [f(x_1)\ f(x_2)\ \cdots\ f(x_n)]$, we obtain

$$g(x) = C[f_1(x)\ f_2(x)\ \cdots\ f_n(x)]^t \quad \text{for } x \in D,$$

which is in $span\{f_1, f_2, \ldots, f_n\}$ such that $g(x_j) = f(x_j)$ for $1 \le j \le n$.

We claim that the above function g is unique. Otherwise suppose that there existed another function $g_0(x) = C_0[f_1(x)\ f_2(x)\ \cdots\ f_n(x)]^t$ in $span\{f_1, f_2, \ldots, f_n\}$ such that $C^t \ne C_0^t \in \mathbb{K}^n$ and $g_0(x_j) = f(x_j)$ for $1 \le j \le n$. Then

$$(C - C_0)[f_1(x_j)\ f_2(x_j)\ \cdots\ f_n(x_j)]^t = g(x_j) - g_0(x_j) = 0 \text{ for } 1 \le j \le n.$$

Since $[f_i(x_j)]_{n\times n}$ is invertible, $C - C_0 = 0$. This contradicts $C \ne C_0$. □

For $n \in \mathbb{N}$, by definition, it is easy to see that the functions $f_i(z) = z^i$ $(0 \le i \le n-1)$ on $\mathbb{C}$ are linearly independent. For any distinct points $z_1, \ldots, z_n$ in $\mathbb{C}$, the determinant of the matrix $[f_i(z_j)]$ is Vandermonde which is nonzero, so $[f_i(z_j)]$ is invertible. Thus the linear independence of $f_i(z) = z^i$ $(0 \le i \le n-1)$ can also be obtained by Theorem 2.1. With this we obtain the following conclusion.

Corollary 2.4. Let $p_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial function with $a_n \neq 0$. Then there exist at most n distinct zeroes of $p_n(z)$.

Proof. Suppose that p_n has $n+1$ distinct zeroes $z_1, z_2, \ldots, z_{n+1}$. Then for

$$Z := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_{n+1} \\ z_1^2 & z_2^2 & \cdots & z_{n+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^n & z_2^n & \cdots & z_{n+1}^n \end{bmatrix}, A^t := \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \text{ and } 0^t := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

we have $AZ = 0$ and $det\, Z \neq 0$ (by Theorem 2.1). Thus $A = 0$, contradicting $a_n \neq 0$. □

Consider the functions $f_1(x) = x^2$ and $f_2(x) = x|x|$ for $x \in \mathbb{R}$ [10]. It is easy to find their Wronskian $W(f_1, f_2)(x) = 0$ for all $x \in \mathbb{R}$ but, by definition or by Theorem 2.1 with $x_1 = -1$ and $x_2 = 1$ (or $x_1 = -1$ and $x_2 = 2$), f_1 and f_2 are linearly independent on $\mathbb{R}$. Note that the set of $x_1, \cdots, x_n$ for $[f_i(x_j)]$ to be invertible is not unique. In addition, the linear independence of $f_1, f_2, \ldots, f_n$ on a set D does not always imply that the matrix $[f_i(x_j)]$ is invertible for all distinct points $x_1, x_2, \ldots, x_n$ in D.

3. Pointwise convergence implies uniform convergence

The title of this section is a statement for a sequence of functions in the space spanned by finite linearly independent functions. Given functions $f_1, \ldots, f_n$ on D, we firstly characterize a pointwise convergent sequence in $span\{f_1, \ldots, f_n\}$ as below.

Theorem 3.1. Let $f_1, \ldots, f_n$ be linearly independent functions from D to $\mathbb{K}$. Suppose that

$$g_k(x) = c_{k1} f_1(x) + c_{k2} f_2(x) + \cdots + c_{kn} f_n(x) \quad \text{for } k \in \mathbb{N} \text{ and } x \in D. \tag{3.1}$$

Then

(*i*) $g_k \to g$ pointwise on D if and only if $c_i := \lim_{k\to+\infty} c_{ki}$ exists for each $1 \leq i \leq n$ and

$$g(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) \quad \text{for all } x \in D. \tag{3.2}$$

(*ii*) If also $f_1, \ldots, f_n$ are bounded on D, then the pointwise convergence of $\{g_k\}$ on D implies the uniform convergence of $\{g_k\}$ on D.

Proof. (*i*) Since $f_1, f_2, \ldots, f_n$ are linearly independent on D, by Theorem 2.1, there exist distinct points $x_1, x_2, \ldots, x_n$ in D such that the matrix $[f_i(x_j)]_{n\times n}$ is invertible.

Suppose that $\{g_k\} \subseteq span\{f_1, \ldots, f_n\}$ is pointwise convergent to g on D. Then

$$g(x) = \lim_{k\to+\infty} g_k(x) \quad \text{for } x \in D$$

and for each $k \in \mathbb{N}$ and each $i = 1, \ldots, n$ there exists c_{ki} in $\mathbb{K}$ such that (3.1) is satisfied for all $x \in D$. In particular,

$$g_k(x_j) = c_{k1} f_1(x_j) + c_{k2} f_2(x_j) + \cdots + c_{kn} f_n(x_j) \quad \text{for } j = 1, \ldots, n,$$

from which we obtain

$$\lim_{k\to+\infty}\begin{bmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{bmatrix}\begin{bmatrix} c_{k1} \\ c_{k2} \\ \vdots \\ c_{kn} \end{bmatrix} = \lim_{k\to+\infty}\begin{bmatrix} g_k(x_1) \\ g_k(x_2) \\ \vdots \\ g_k(x_n) \end{bmatrix} = \begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_n) \end{bmatrix}.$$

In addition,

$$\begin{bmatrix} c_{k1} \\ c_{k2} \\ \vdots \\ c_{kn} \end{bmatrix} \to \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} := \begin{bmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{bmatrix}^{-1}\begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_n) \end{bmatrix} \quad \text{as } k\to+\infty.$$

Thus g satisfies (3.2) for all $x \in D$.

Conversely, if for each $k \in \mathbb{N}$ there exists $(c_{k1}, c_{k2}, \ldots, c_{kn})^t \in \mathbb{K}^n$ such that each $c_i := \lim_{k\to+\infty} c_{ki}$ exists for $1 \le i \le n$ with (3.1) and (3.2) being satisfied for all $x \in D$, then it is easy to see that $g_k \to g$ pointwise on D.

(ii) Now, suppose that $f_1, \ldots, f_n$ are bounded on D. Since

$$\begin{aligned} |g_k(x) - g(x)| &= \left|\sum_{i=1}^n (c_{ki} - c_i) f_i(x)\right| \le \sum_{i=1}^n |c_{ki} - c_i||f_i(x)| \\ &\le \sum_{i=1}^n |c_{ki} - c_i| \sup\{|f_i(x)| : x \in D\} \\ &\le \max_{1\le i\le n}\{\sup\{|f_i(x)| : x \in D\}\} \sum_{i=1}^n |c_{ki} - c_i| \quad \text{for all } x \in D \end{aligned}$$

and $\lim_{k\to+\infty} c_{ki} = c_i$ for $1 \le i \le n$,

$$\lim_{k\to+\infty} \sup\{|g_k(x) - g(x)| : x \in D\} = 0.$$

It follows that $\{g_k\}$ is uniformly convergent on D. □

Remark 1. For functions $f_1, \ldots, f_n$ in Theorem 3.1, if there exist functions $e_1, \ldots, e_n$ on D and $d_{i1}, \ldots, d_{in}$ in $\mathbb{K}$ such that

$$f_i(x) = d_{i1}e_1(x) + d_{i2}e_2(x) + \cdots + d_{in}e_n(x) \quad \text{for } 1 \le i \le n \text{ and } x \in D,$$

then, by Theorem 2.2, $e_1, e_2, \ldots, e_n$ are linearly independent.

For $\{g_k\} \subseteq span\{f_1, \ldots, f_n\}$, since for each $k \in \mathbb{N}$ the coordinates $c_{k1}, c_{k2}, \ldots, c_{kn}$ of g_k in $span\{f_1, \ldots, f_n\}$ satisfy

$$\begin{aligned} g_k(x) &= c_{k1}f_1(x) + c_{k2}f_2(x) + \cdots + c_{kn}f_n(x) \\ &= \sum_{i=1}^n c_{ki} \sum_{j=1}^n d_{ij}e_j(x) = \sum_{j=1}^n \left(\sum_{i=1}^n c_{ki}d_{ij}\right) e_j(x) \quad \text{for } x \in D, \end{aligned}$$

it follows from Theorem 3.1 that $g_k \to g$ pointwise on D if and only if, for the coordinates $c_{k1}, c_{k2}, \ldots, c_{kn}$ of g_ks, $\lim_{k\to+\infty} \sum_{i=1}^n c_{ki}d_{ij}$ exists for all $1 \le j \le n$. So the pointwise convergence of $\{g_k\}$ is independent of the bases of $span\{f_1, \ldots, f_n\}$.

Next, we apply Theorem 3.1 to study the continuity, differentiability, and integrability for the limit function of a pointwise convergent sequence in $span\{f_1, \ldots, f_n\}$.

Theorem 3.2. Let D be a subset in a metric space, $f_1, \dots, f_n$ be linearly independent functions on D, and $\{g_k\} \subseteq span\{f_1, \dots, f_n\}$. Suppose that $g_k \to g$ pointwise.

(*i*) If x is a limit point of D, $\lim_{t\to x} f_i(t)$ exists, and f_i is bounded for $1 \le i \le n$, then $\lim_{t\to x} g_k(t)$ exists and

$$\lim_{t\to x}\lim_{k\to+\infty} g_k(t) = \lim_{t\to x} g(t) = \lim_{k\to+\infty}\lim_{t\to x} g_k(t).$$

(*ii*) If $D = [a, b]$ and $f_1, \dots, f_n$ are differentiable at $x \in [a, b]$, then g_k is differentiable at x for $k \in \mathbb{N}$, $\{g'_k(x)\}$ is convergent and $\lim_{k\to+\infty} g'_k(x) = g'(x)$.

(*iii*) If $D = [a, b]$ and $f_1, \dots, f_n$ are integrable on $[a, b]$, then, for $k \in \mathbb{N}$, g_k is integrable on $[a, b]$ and, for $x \in (a, b]$, $\{\int_a^x g_k\}$ is convergent and

$$\lim_{k\to+\infty}\int_a^x g_k = \int_a^x g.$$

Proof. For $k \in \mathbb{N}$ and g_k in $span\{f_1, \dots, f_n\}$, we have $c_{ki} \in \mathbb{K}$ $(1 \le i \le n)$ such that

$$g_k(x) = c_{k1}f_1(x) + c_{k2}f_2(x) + \cdots + c_{kn}f_n(x) \quad \text{for all } x \in D. \tag{3.3}$$

Since $f_1, \dots, f_n$ are linearly independent on D and $g_k \to g$ pointwise, by Theorem 3.1, $\lim_{k\to+\infty} c_{ki} = c_i \in \mathbb{K}$ for $1 \le i \le n$, and

$$g(x) = \lim_{k\to+\infty} g_k(x) = c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) \quad \text{for all } x \in D. \tag{3.4}$$

Now, if f_i is bounded for $1 \le i \le n$, by Theorem 3.1 again, $g_k \to g$ uniformly.

For a limit point x of D, if $\lim_{t\to x} f_i(t)$ exists for $1 \le i \le n$, then, for each $k \in \mathbb{N}$, $A_k := \lim_{t\to x} g_k(t)$ exists. By [12, Theorem 7.11], $\{A_k\}$ converges, and $\lim_{t\to x} g(t) = \lim_{k\to+\infty} A_k$. Hence (*i*) follows.

To show (*ii*), let $D = [a, b]$ and $f_1, \dots, f_n$ be differentiable at $x \in [a, b]$. Then g_k is differentiable at x and, by (3.3),

$$g'_k(x) = c_{k1}f'_1(x) + c_{k2}f'_2(x) + \cdots + c_{kn}f'_n(x) \quad \text{for } k \in \mathbb{N}.$$

This with $\lim_{k\to+\infty} c_{ki} = c_i \in \mathbb{K}$ for $1 \le i \le n$ and (3.4) implies that $\{g'_k(x)\}$ satisfies

$$\begin{aligned}\lim_{k\to+\infty} g'_k(x) &= \lim_{k\to+\infty}[c_{k1}f'_1(x) + c_{k2}f'_2(x) + \cdots + c_{kn}f'_n(x)] \\ &= c_1f'_1(x) + c_2f'_2(x) + \cdots + c_nf'_n(x) = g'(x).\end{aligned}$$

Next, suppose that $f_1, \dots, f_n$ are integrable on $D = [a, b]$. By (3.3), g_k is integrable on $[a, b]$ and, for $k \in \mathbb{N}$ and $x \in (a, b]$,

$$\int_a^x g_k = c_{k1}\int_a^x f_1 + c_{k2}\int_a^x f_2 + \cdots + c_{kn}\int_a^x f_n.$$

It follows from $\lim_{k\to+\infty} c_{ki} = c_i \in \mathbb{K}$ for $1 \le i \le n$ and (3.4) that $\{\int_a^x g_k\}$ is convergent and $\lim_{k\to+\infty}\int_a^x g_k = \int_a^x g$. Thus (*iii*) is valid. □

Remark 2. For linearly independent functions $f_1, \dots, f_n$ on $[a, b]$, if they are continuous on $[a, b]$, then it is easy to see that $\int_a^x f_1, \dots, \int_a^x f_n$ are linearly independent on $[a, b]$. However, even if they are differentiable in (a, b), $f'_1, \dots, f'_n$ may be linearly dependent in (a, b). For example, the functions $1, x, \dots, x^{n-1}$ are linearly independent and differentiable in $(0, 1)$, but their derivatives $0, 1, \dots, x^{n-2}$ are not linearly independent in $(0, 1)$.

When functions $f_1, f_2, \ldots, f_n$ are continuous, the uniform convergence of a sequence in $span\{f_1, \ldots, f_n\}$ can also be characterized as below.

Theorem 3.3. Let D be a subset in a metric space, functions $f_1, \ldots, f_n : D \to \mathbb{K}$ be continuous, bounded and linearly independent on D, and $\{g_k\} \subseteq span\{f_1, \ldots, f_n\}$. Then the following statements are equivalent:

(i) $\{g_k\}$ is pointwise convergent on D.
(ii) $\{g_k\}$ is uniformly convergent on D.
(iii) There exists $g : D \to \mathbb{K}$ such that for each $x \in D$ and each sequence $\{x_k\}$ in D converging to x, there holds $\lim_{k\to+\infty} g_k(x_k) = g(x)$.

Proof. Since $(i) \Rightarrow (ii)$ follows directly from Theorem 3.1 and $(iii) \Rightarrow (i)$ is obvious, it suffices to show $(ii) \Rightarrow (iii)$.

Suppose that g_k converges uniformly to g. Since $f_1, \cdots, f_n$ are continuous, by Theorem 3.2, g is continuous. For $x \in D$, if $\{x_k\}$ is a sequence in D converging to x, then

$$|g_k(x_k) - g(x_k)| \leq \sup\{|g_k(u) - g(u)| : u \in D\}.$$

Since $\sup\{|g_k(u) - g(u)| : u \in D\} \to 0$ as $k \to +\infty$, it follows that

$$\lim_{k\to+\infty} g_k(x_k) = \lim_{k\to+\infty} g(x_k) = g(x).$$

Thus (iii) is valid. □

If $\{g_k\}$ is not in a finite dimensional space of functions and it is only pointwise convergent but not uniformly convergent, then there may exist $x \in D$ such that $\{g_k(x_k)\}$ does not converge even $x_k \to x$ as $k \to +\infty$.

Example 3.1. *Consider $\{g_k\} \subseteq C([0,1])$ defined by $g_k(x) = x^k$ for $x \in [0,1]$. For the sequence $\{x_k\}$ given by*

$$x_k = 1 - \frac{1}{k} \text{ for } k = 2m \text{ and } x_k = 1 - \frac{2}{k} \text{ for } k = 2m+1 \text{ with } m \in \mathbb{N},$$

it is easy to see that $x_k \to 1$ as $k \to +\infty$. However,

$$\lim_{m\to+\infty} g_{2m}(x_{2m}) = e^{-1} \text{ and } \lim_{m\to+\infty} g_{2m+1}(x_{2m+1}) = e^{-2},$$

that is, $\lim_{k\to+\infty} g_k(x_k)$ does not exist.

Example 3.2. *Consider a sequence $\{g_k\}$, where $g_k : \mathbb{N} \to \mathbb{R}$ for $k \in \mathbb{N}$. Suppose that for each $i \in \mathbb{N}$, $g_k(i) \to a_i$ as $k \to +\infty$ and $a_i \to a$ as $i \to +\infty$. The case $\lim_{k\to+\infty} g_k(k) \neq a$ may happen. For example, let*

$$g_k(i) = \left(1 - \frac{1}{i}\right)^k \qquad \text{for } i, k \in \mathbb{N}.$$

For each $i \in \mathbb{N}$, we have $\lim_{k\to+\infty} g_k(i) = 0 =: a_i$ and $\lim_{i\to+\infty} a_i = 0$ but

$$\lim_{k\to+\infty} g_k(k) = \lim_{k\to+\infty} (1 - \frac{1}{k})^k = e^{-1} \neq 0.$$

4. What a convergent sequence of initial conditions means

In the last section, we consider the following initial value problem:

$$f^{(n)}(x) + a_{n-1}f^{(n-1)}(x) + \cdots + a_1 f'(x) + a_0 f(x) = 0 \tag{4.1}$$

$$\text{subject to} \quad [f(0)\ f'(0)\ \cdots\ f^{(n-1)}(0)]^t = B := [b_0\ b_1\ \cdots\ b_{n-1}]^t, \tag{4.2}$$

where a_is and b_is are constants in $\mathbb{R}$ for $0 \le i \le n-1$. For the initial value problem,

(i) if function f is a solution of (4.1) on an interval I, then $f, f', \dots, f^{(n)}$ are linearly dependent on I, so for any distinct points $x_1, x_2, \dots, x_{n+1}$ in I,

$$det[f^{(i)}(x_j)]_{(n+1)\times(n+1)} = 0,$$

where $f^{(0)}(x) = f(x)$.

(ii) By Theorem 2.1, $f, f', \dots, f^{(n-1)}$ are linearly independent in I iff there exist distinct points $x_1, x_2, \dots, x_n$ in I such that $[f^{(i)}(x_j)]_{n\times n}$ is invertible.

(iii) For solution functions $f_1, f_2, \dots, f_n$ of (4.1), they are linearly independent in I iff the Wronskian $W(f_1, f_2, \dots, f_n)(x) \neq 0$ for all x in I iff there exist $x_1, x_2, \dots, x_n$ in I such that $[f_i(x_j)]$ is invertible.

Given n linearly independent solution functions $g_1, g_2, \dots, g_n$ of (4.1), for any $c_1, c_2, \dots, c_n$ in $\mathbb{R}$, the function

$$f(x) = c_1 g_1(x) + c_2 g_2(x) + \cdots + c_n g_n(x)$$

is a solution of (4.1) completely determined by its initial condition when its derivatives at 0 in (4.2) are available. In this section we further demonstrate that f can also be determined by its values at n appropriate points without using derivatives.

Theorem 4.1. Let $g_1, g_2, \dots, g_n$ be linearly independent solutions of (4.1). Then for each solution f of (4.1) and each closed interval $[a, b]$ satisfying $0 \in [a, b] \neq \{0\}$, there exist $c_1, c_2, \dots, c_n$ in $\mathbb{R}$ and distinct points $x_1, x_2, \dots, x_n$ in $[a, b]$ such that

$$f(x) = c_1 g_1(x) + c_2 g_2(x) + \cdots + c_n g_n(x) \quad \text{for } x \in [a, b] \tag{4.3}$$

$$\text{and} \quad \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} g_1(x_1) & g_2(x_1) & \dots & g_n(x_1) \\ g_1(x_2) & g_2(x_2) & \dots & g_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(x_n) & g_2(x_n) & \dots & g_n(x_n) \end{bmatrix}^{-1} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix} \tag{4.4}$$

$$= \begin{bmatrix} g_1(0) & g_2(0) & \cdots & g_n(0) \\ g_1'(0) & g_2'(0) & \cdots & g_n'(0) \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{(n-1)}(0) & g_2^{(n-1)}(0) & \cdots & g_n^{(n-1)}(0) \end{bmatrix}^{-1} \begin{bmatrix} f(0) \\ f'(0) \\ \vdots \\ f^{(n-1)}(0) \end{bmatrix}. \tag{4.5}$$

Proof. Since $g_1, g_2, \dots, g_n$ are linearly independent solutions of (4.1), the general solution f of (4.1) is given by (4.3) and for each closed interval $[a, b]$ satisfying $0 \in [a, b] \neq \{0\}$, by Theorem 2.1, there are distinct points $x_1, x_2, \dots, x_n$ in $[a, b]$ such that

$$[g_i(x_j)] := \begin{bmatrix} g_1(x_1) & g_2(x_1) & \dots & g_n(x_1) \\ g_1(x_2) & g_2(x_2) & \dots & g_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(x_n) & g_2(x_n) & \dots & g_n(x_n) \end{bmatrix} \tag{4.6}$$

is invertible. It follows from (4.3) that

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} g_1(x_1) & g_2(x_1) & \dots & g_n(x_1) \\ g_1(x_2) & g_2(x_2) & \dots & g_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(x_n) & g_2(x_n) & \dots & g_n(x_n) \end{bmatrix}^{-1} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}.$$

On the other hand, (4.3) implies $f^{(k)}(x) = \sum_{i=1}^{n} c_i g_i^{(k)}(x)$ $(0 \le k \le n-1)$, so

$$\begin{bmatrix} f(0) \\ f'(0) \\ \vdots \\ f^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} g_1(0) & g_2(0) & \cdots & g_n(0) \\ g_1'(0) & g_2'(0) & \cdots & g_n'(0) \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{(n-1)}(0) & g_2^{(n-1)}(0) & \cdots & g_n^{(n-1)}(0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

from which we obtain

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} g_1(0) & g_2(0) & \cdots & g_n(0) \\ g_1'(0) & g_2'(0) & \cdots & g_n'(0) \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{(n-1)}(0) & g_2^{(n-1)}(0) & \cdots & g_n^{(n-1)}(0) \end{bmatrix}^{-1} \begin{bmatrix} f(0) \\ f'(0) \\ \vdots \\ f^{(n-1)}(0) \end{bmatrix}.$$

The proof is complete. □

From Theorem 4.1, the linearly independent solutions of (4.1) can also be further characterized in terms of their initial conditions.

Theorem 4.2. Let $g_1, g_2, \ldots, g_n$ be linearly independent solutions of (4.1) and let $f_i(x) = c_{i1}g_1(x) + c_{i2}g_2(x) + \cdots + c_{in}g_n(x)$ and $c_{ij} \in \mathbb{K}$ for $1 \le i, j \le n$. Then the following statements are equivalent:

(*i*) $f_1, f_2, \ldots, f_n$ are linearly independent on $[a, b]$ with $0 \in [a, b] \neq \{0\}$.

(*ii*) For $[a, b]$ with $0 \in [a, b] \neq \{0\}$, there exist $x_1, x_2, \ldots, x_n$ in $[a, b]$ such that $[f_i(x_j)]$ is invertible.

(*iii*) $[f_i^{(j-1)}(0)]$ is invertible.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ is immediate from Theorem 2.1, so it suffices to show $(ii) \Leftrightarrow (iii)$.

Since $g_1, g_2, \ldots, g_n$ are linearly independent, for each $[a, b]$ with $0 \in [a, b] \neq \{0\}$, by Theorem 2.1, there exist distinct points $x_1, x_2, \ldots, x_n$ in $[a, b]$ such that $G := [g_i(x_j)]$ is invertible.

For convenience, we denote

$$F := [f_i(x_j)], \quad F_0^{(n-1)}(0) := [f_i^{(j-1)}(0)], \quad G_0^{(n-1)}(0) := [g_i^{(j-1)}(0)].$$

For $1 \le i \le n$, since $f_i(x) = c_{i1}g_1(x) + c_{i2}g_2(x) + \cdots + c_{in}g_n(x)$,

$$f_i(x_j) = \sum_{k=1}^{n} c_{ik} g_k(x_j) \text{ and } f_i^{(j-1)}(0) = \sum_{k=1}^{n} c_{ik} g_k^{(j-1)}(0) \text{ for } 1 \le j \le n.$$

Note that G and $G_0^{(n-1)}(0)$ are both invertible and

$$FG^{-1} = [f_i(x_j)][g_i(x_j)]^{-1} = [c_{ij}] = F_0^{(n-1)}(0)[G_0^{(n-1)}(0)]^{-1}.$$

It follows that

$$F = F_0^{(n-1)}(0)[G_0^{(n-1)}(0)]^{-1}G \quad \text{and} \quad F_0^{(n-1)}(0) = FG^{-1}G_0^{(n-1)}(0).$$

Thus $[f_i(x_j)]$ is invertible iff $[f_i^{(j-1)}(0)]$ is invertible, that is, $(ii) \Leftrightarrow (iii)$. □

Denote a sequence of initial conditions of (4.2) by $\{B_k\}$, where

$$B_k = [b_{0k}\ b_{1k}\ \cdots\ b_{(n-1)k}]^t \quad \text{for } k \in \mathbb{N}.$$

It is known that equation (4.1) has n linearly independent solutions which are bounded on each closed interval interval containing 0. So Theorem 3.1 is applicable here. We will use Theorems 2.1, 3.1, and 3.2 to show that the (pointwise or uniform) convergence of a sequence of solutions of (4.1) is equivalent to that of the sequence of relevant initial conditions.

Theorem 4.3. Given a sequence $\{B_k\}$ of initial conditions of (4.2), let f_k be the solution of (4.1) subject to $[f_k(0)\ f_k'(0)\ \cdots\ f_k^{(n-1)}(0)]^t = B_k$ for $k \in \mathbb{N}$. Then the following statements are equivalent:

- (i) $\{B_k\}$ converges.
- (ii) $\{f_k\}$ converges pointwise.
- (iii) All $\{f_k\}, \{f_k'\}, \dots, \{f_k^{(n-1)}\}$ converge pointwise.
- (iv) All $\{f_k\}, \{f_k'\}, \dots, \{f_k^{(n-1)}\}$ converge uniformly on each closed interval $[a, b]$ satisfying $0 \in [a, b] \neq \{0\}$.
- (v) $\{f_k\}$ converges uniformly on each closed interval $[a, b]$ satisfying $0 \in [a, b] \neq \{0\}$.

Proof. $(i) \Rightarrow (ii)$. As we know, there exist n linearly independent solutions $g_1, g_2, \dots, g_n$ of (4.1) and the general solution f of (4.1) is

$$f(x) = c_1 g_1(x) + c_2 g_2(x) + \cdots + c_n g_n(x), \text{where } c_j \in \mathbb{R} \text{ for } 1 \leq j \leq n.$$

For any $x \in \mathbb{R}$ and closed interval $[a, b]$ containing 0 and x, by Theorem 2.1, there exist distinct points $x_1, x_2, \dots, x_n$ in $[a, b]$ such that

$$[g_i(x_j)] := \begin{bmatrix} g_1(x_1) & g_2(x_1) & \dots & g_n(x_1) \\ g_1(x_2) & g_2(x_2) & \dots & g_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(x_n) & g_2(x_n) & \dots & g_n(x_n) \end{bmatrix} \tag{4.7}$$

is invertible.

For each $k \in \mathbb{N}$, the solution of (4.1) subject to

$$[f_k(0)\ f_k'(0)\ \cdots\ f_k^{(n-1)}(0)]^t = B_k$$

is $f_k(x) = \sum_{j=1}^n c_{kj} g_j(x)$. From this we have $B_k = AC_k$, where

$$A := \begin{bmatrix} g_1(0) & g_2(0) & \dots & g_n(0) \\ g_1'(0) & g_2'(0) & \dots & g_n'(0) \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{(n-1)}(0) & g_2^{(n-1)}(0) & \dots & g_n^{(n-1)}(0) \end{bmatrix} \text{ is invertible and } \ C_k := \begin{bmatrix} c_{k1} \\ c_{k2} \\ \vdots \\ c_{kn} \end{bmatrix},$$

so $f_k(x) = \sum_{j=1}^n c_{kj} g_j(x) = [g_1(x)\ g_2(x)\ \cdots\ g_n(x)] A^{-1} B_k$. And hence, if $\{B_k\}$ converges, then $\{f_k(x)\}$ converges. Thus (ii) follows.

The implications $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are immediate from Theorems 3.2 and 3.1 (and their proofs) respectively since all $\{f_k\}, \{f_k'\}, \dots, \{f_k^{(n-1)}\}$ are continuous on each closed interval $[a, b]$ satisfying $0 \in [a, b] \neq \{0\}$.

$(iv) \Rightarrow (v)$ is obvious while the implication $(v) \Rightarrow (i)$ can be proved by $(v) \Rightarrow (iv) \Rightarrow (i)$, which is immediate from Theorem 3.2 and (iv) with $x = 0$. Thus the proof is complete. □

Remark 3. The implication $(v) \Rightarrow (i)$ in Theorem 4.3 can also be directly derived as below: Let $[a, b]$ be a closed interval containing $0, x_1, x_2, \ldots, x_n$ and let $f_k \to f$ uniformly on $[a, b]$. Then

$$\lim_{k\to+\infty} f_k(x_i) = f(x_i) \quad \text{for } 1 \le i \le n$$

and, by Theorem 3.1, there exists $C = [c_1\ c_2\ \cdots\ c_n]^t$ such that $f(x) = \sum_{i=1}^n c_j g_j(x)$, from which we have

$$\begin{bmatrix} g_1(x_1) & g_2(x_1) & \ldots & g_n(x_1) \\ g_1(x_2) & g_2(x_2) & \ldots & g_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(x_n) & g_2(x_n) & \ldots & g_n(x_n) \end{bmatrix}^{-1} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

This with

$$\begin{bmatrix} g_1(x_1) & g_2(x_1) & \ldots & g_n(x_1) \\ g_1(x_2) & g_2(x_2) & \ldots & g_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ g_1(x_n) & g_2(x_n) & \ldots & g_n(x_n) \end{bmatrix}^{-1} \begin{bmatrix} f_k(x_1) \\ f_k(x_2) \\ \vdots \\ f_k(x_n) \end{bmatrix} = \begin{bmatrix} c_{k1} \\ c_{k2} \\ \vdots \\ c_{kn} \end{bmatrix} \quad \text{for } k \in \mathbb{N}$$

implies that $c_k = \lim_{k\to+\infty} c_{kj}$ for $1 \le j \le n$.

Now, for $B_k = [b_{0k}\ b_{1k}\ \cdots\ b_{(n-1)k}]^t$, since $B_k = AC_k$,

$$\lim_{k\to+\infty} B_k = \lim_{k\to+\infty} AC_k = AC.$$

This shows that (i) is valid.

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