Nonlinear wave diffraction by an uneven viscoelastic seafloor

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This abstract introduces an extension to the study on wave interaction with an elastic sheet on a viscoelastic foundation which we presented at the previous workshop [1]. Here we propose to investigate nonlinear wave interaction with a deformable seafloor lying on an uneven bottom topography, representing the submerged shelf, ramp or periodic ripples formed by tidal waves on the ocean floor. Muddy sea regions are commonly observed along the shores in the coastal areas where water depth can vary significantly [2, 3]. Therefore, this study is a step forward to a more realistic models of wave damping by deformable seafloors. Many mathematical theories have been proposed in the past, treating the soft and movable seabeds as a highly-viscous fluid with various complemetary properties including elasticity, plasticity and porosity [4, 5, 6]. Alternatively, the fluid-bottom interface can be replaced by a thin elastic plate and the restoring force of mud can be modelled by the action of spring and dampers. This approach, first proposed a decade ago [7, 8], remains mostly unexplored in spite of its wide applicability to other problems like wave interaction with floating ice or large elastic structures. This study is an attempt to contribute to the development of this model as applied to the problems of wave-mud interaction.

STATEMENT OF THE PROBLEM

The equations governing the motion of the fluid are provided by the nonlinear Green-Naghdi theory (GN hereafter), originally developed from the theory of directed fluid sheets [9]. This theory can be applied to a wide class of problems, including wave interaction with floating structures and propagation of water waves over an arbitrary seafloor, see e.g. [10]. The GN equations are classified based on the level of the functions used to prescribe the distribution of the vertical velocity along the water column. In the Level I GN theory, also known as the restricted theory, the vertical component of fluid particle velocity is a linear function of the vertical coordinate, which results in the horizontal velocities being invariant along the depth of the fluid. This single assumption makes the Level I GN equations feasible for description of fairly long waves or waves in shallow waters.

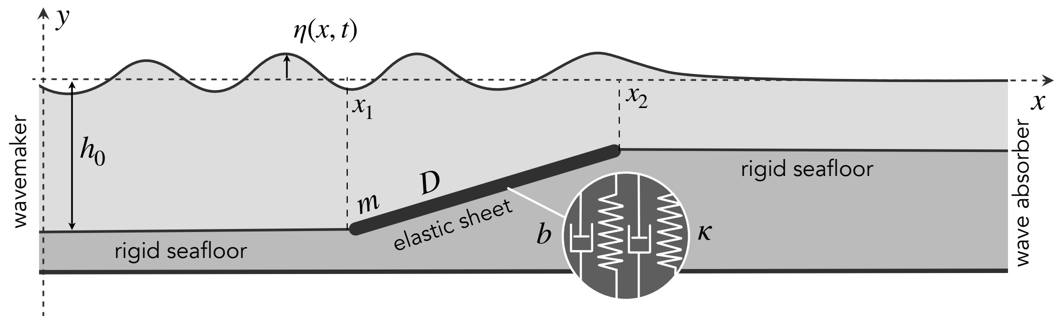

Figure 1: Schematics of wave interaction with a deformable elastic sheet lying on a viscoelastic foundation.

The flow of incompressible and inviscid fluid is considered in a two-dimensional Cartesian reference frame in which the x axis is pointing to the right, the y axis is directed upward, and its origin is located on the undisturbed free surface of the fluid (see figure 1). The governing equations are given by mass and momentum conservation laws in dimensionless form after using the density of the fluid ρ, the fluid depth at the wavemaker h_0 and acceleration due to gravity g as a dimensionally independent set:

$$\eta_{,t} + [(\eta - h - \alpha)u]_{,x} = \alpha_{,t}, \tag{1}$$

$$u_{,t} + uu_{,x} + \eta_{,x} = -\frac{1}{6}\Big\{(2\eta+\alpha)_{,x}\ddot{\alpha} + (4\eta-\alpha)_{,x}\ddot{\eta} + (\eta-h-\alpha)(\ddot{\alpha}+2\ddot{\eta})_{,x}\Big\}. \tag{2}$$

Here $u(x,t)$ is the depth-averaged horizontal fluid velocity, $\eta(x,t)$ is the free surface elevation measured from the still-water level, $\alpha(x,t)$ is the deformation of the bottom measured from its stationary position, $h(x)$ is a predetermined partly-smoothed function describing the bottom topography. In our notation, subscript after comma denotes the partial differentiation with respect to the given variable and superimposed dot specifies the total time derivative. In this study, an uneven rigid bottom is partly covered by an elastic sheet which is supported by a uniform system of springs and dampers. The condition of balance of hydrodynamic pressure predicted by the GN theory on the bottom and elastic pressure induced by deformation of the elastic sheet, complements the system of mass and momentum equations (1)-(2):

$$\frac{1}{2}(\eta - h - \alpha)(\ddot{\alpha} + \ddot{\eta}) + \eta - \alpha + m\alpha_{,tt} + D\alpha_{,xxxx} + \kappa\alpha + b\alpha_{,t} = 0. \tag{3}$$

Here, m and D are the unit mass and flexural rigidity of the sheet, b and κ are the viscous damping and stiffness of the elastic foundation per unit area, respectivey.

A piecewise-smooth function of a variable topography can be chosen arbitrarily and represents either a segment of an inclined bottom, or a wavy bottom-surface, or a protruding shelf ($n = 1$):

$$h(x) = \begin{cases} -1, & x < x_1, \\ -1 + ax, & x_1 \le x \le x_2, \\ -1 + a(x_2 - x_1), & x > x_2, \end{cases} \qquad h(x) = \begin{cases} -1, & x < x_1, \\ -1 + a\sin\dfrac{\pi n(x - x_1)}{x_2 - x_1}, & x_1 \le x \le x_2, \\ -1, & x > x_2. \end{cases} \tag{4}$$

By virtue of the GN equations, the exact nonlinear kinematic and dynamic boundary conditions at the free surface and the impermeability condition on the bottom, are already satisfied by the system of equations (1) and (2). In order to construct the solution, valid in the entire fluid domain, the appropriate jump and matching conditions should be imposed across the lines ($x = x_1$, $x = x_2$), dividing the regions with rigid and deformable bottoms. Motivated by the physics of the problem, we require the continuity of the free surface elevation and fluid pressure on the seafloor, predicted by the GN theory (see [10]):

$$\Big[\eta\Big] = 0, \quad \Big[\frac{1}{2}(\eta - h - \alpha)(\ddot{\eta} + \ddot{\alpha} + 2)\Big] = 0. \qquad (x = x_1, \quad x = x_2) \tag{5}$$

The elastic sheet on the bottom is assumed to be fixed at the edges and hence the deformations, bending moments and shear stresses should vanish at the discontinuity curves:

$$\alpha = 0, \quad D\alpha_{,xx} = 0, \quad D\alpha_{,xxx} = 0. \qquad (x = x_1, \quad x = x_2) \tag{6}$$

METHOD OF SOLUTION

The set of equations (1)-(3) for the fluid flow above an uneven and deformable seafloor, complemented by boundary conditions (5)-(6), are solved simultaneously in the entire flow domain. The numerical solution is found with the use of a finite-difference technique and the modified Euler's time-stepping method. In our previous study [1], we developed an approach to deal with the system of equations (1), (2) and (3) numerically, when the wave propagates over an infinite deformable sheet lying on the even bottom, i.e. $h(x) \equiv 0$. By an appropriate transformation of the system (1)-(3), the second-order full time derivatives of the free surface elevation, can be eliminated from the momentum equation (2) and pressure balance equation (3). As a result, the equations containing the time derivativies of the unknown horizontal velocity $u(x,t)$ and the deformations of the bottom $\alpha(x,t)$ can be derived. In the case of an uneven bottom, the resulting equations include spatial derivatives of the given function $h(x)$ and attain the following form, suitable for numerical modeling:

$$\begin{aligned}(1 + (h_{,x} + \alpha_{,x})\eta_{,x} + \frac{\phi}{2}(h_{,xx} + \alpha_{,xx}))u_{,t} - \phi\phi_{,x}u_{,xt} - \frac{\phi^2}{3}u_{,xxt} + \\ +(\phi u_{,x} + 2\eta_{,x}u)\alpha_{,xt} + \phi u\alpha_{,xxt} + \eta_{,x}\alpha_{,tt} + \frac{\phi}{2}\alpha_{,xtt} = -\overline{Y},\end{aligned} \tag{7}$$

$$\phi(h_{,x} + \alpha_{,x})u_{,t} - \frac{1}{2}\phi^2 u_{,xt} + b\alpha_{,t} + 2\phi u\alpha_{,xt} + (\phi + m)\alpha_{,tt} = -\underline{Y}, \tag{8}$$

where functions $\overline{Y}$ and $\underline{Y}$, including only the spatial derivatives of the unknown functions, account for the effects of the free surface and deformable bottom, respectively:

$$\overline{Y} = \eta_{,x} + (\eta_{,x}[h_{,xx} + \alpha_{,xx}] + \frac{1}{2}\phi[h_{,xxx} + \alpha_{,xxx}])u^2 + (1 + [h_{,x} + \alpha_{,x}]\eta_{,x} + \frac{3}{2}\phi[h_{,xx} + \alpha_{,xx}])uu_{,x} + \\ + \frac{1}{3}\phi^2(u_{,x}u_{,xx} - uu_{,xxx}) + \phi\eta_{,x}(u_{,x}^2 - uu_{,xx}) + \frac{1}{2}\phi(h_{,x} + \alpha_{,x})(u_{,x}^2 + uu_{,xx}), \tag{9}$$

$$\underline{Y} = \eta - \alpha + k\alpha + D\alpha_{,xxxx} + \frac{1}{2}\phi^2(u_{,x}^2 - uu_{,xx}) + \phi u(\alpha_{,x}u_{,x} - u\alpha_{,xx}). \tag{10}$$

Here $\phi = \eta - h - \alpha$ denotes the thickness of the fluid sheet. The system (7)-(8) is solved numerically by use of the iterative method and Gaussian elimination algorithm at each iteration. Once the horizontal velocity u and bottom deformation α are calculated, the free surface elevation η on each time step is obtained from the mass continuity equation (1).

WAVE DIFFRACTION

The hydrodynamic pressure force induced by the travelling wave on the free surface causes the deflection of the sheet on the bottom and invokes the dynamic response of the attached springs and dampers. Due to the interaction with the deformable seafloor, the free surface wave should experience significant diffraction and attenuation. Two factors have an effect on the wave transformation: deformations of the viscoelastic seafloor and unevenness of the rigid bottom. In this abstract, the uneven bottom is represented by a constant slope starting at the wavemaker, specified by a constant a in equation (4). The discussion of other types of bottom rheology will be presented at the workshop.

Fig. 2 compares the profiles of a solitary wave at a fixed moment of time for different bottom conditions. Compared to the rigid bottom case, there is a considerable decrease of the wave amplitude and propagation speed resulting from the energy exchange between the fluid flow and the deformable seafloor. The wave looses its energy gradually with propagated distance by the work done on deflecting the seafloor, which by turn creates a constantly changing disturbance for the wave flow. The formation of a secondary wave in the wake of the wave front can be also attributed to the elastic deformations of the seafloor. The presence of a slope compensates for the decrease of the wave amplitude and further reduces the wave propagation speed.

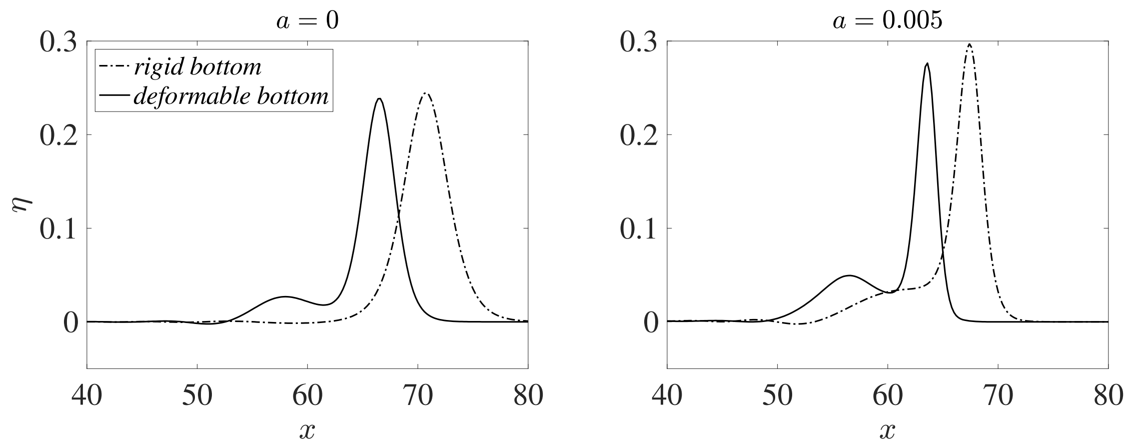

Figure 2: Snapshots of free surface elevation at time $t = 100$ for a solitary wave ($A = 0.25$) propagating over a rigid and deformable ($m = 0.05$, $D = 0.1$, $\kappa = 5$, $b = 0.05$) seafloors with different inclination parameter a.

For periodic waves, the change of propagation speed reveals itself in modulation of the wavelength, which is illustrated in Figs. 3 and 4. Fig. 3 shows snapshots of the free surface for the cnoidal wave propagating over the rigid and deformable seafloors with and without inclination angle. The observed decrease of the wavelength caused by deformations of the seafloor is intensified by the presence of a slope. The wave propagating above the sloping seafloor is steeper as compared to that above the flat seafloor, in both rigid and deformable configurations. The resulting pattern of the wave trasnformation shall depend on different factors, including mass and rigidity of the elastic sheet, stiffness and damping coefficients of the viscoelastic foundation, parameters of an uneven bottom, as well as the incoming free surface wavelength and the wave height. Fig. 4 shows the ratio of the diffracted wavelength, λ^*, to the incoming wavelength, λ, against stiffness and damping coefficients of the viscoelastic foundation with and without a slope. The mutual effects of the bottom inclination angle and the properties of the viscoelastic foundation on the wave diffraction characteristics are demonstrated to be minor. As shown in Fig. 4, the wavelength is extremely sensitive to the variations of the stiffness and damping parameters and decreases significantly as stiffness drops to the critical value.

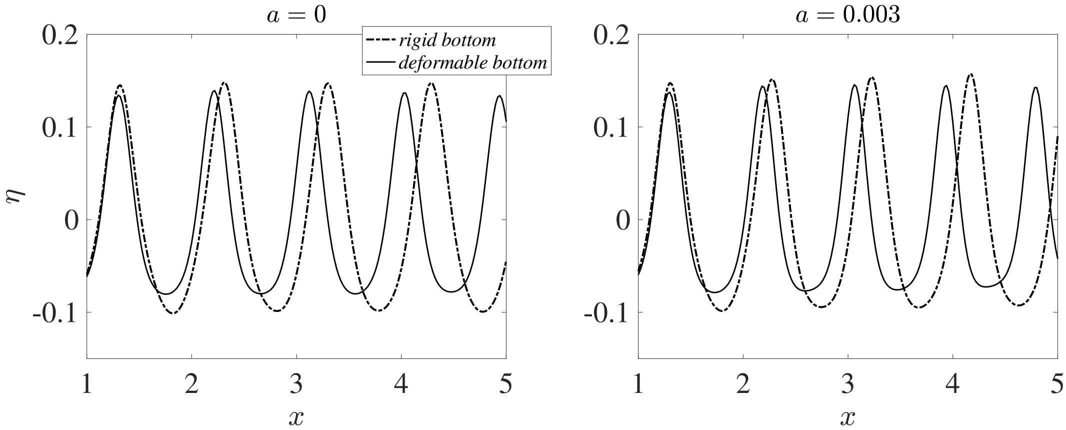

Figure 3: Snapshots of (a) free surface elevation and (b) bottom displacement at time $t = 125$ for a cnoidal wave ($H = 0.25$, $\lambda = 10$) propagating over a rigid and deformable ($m = 0.05$, $D = 0.1$, $\kappa = 3$, $b = 0.2$) seafloors with different inclination parameter a.

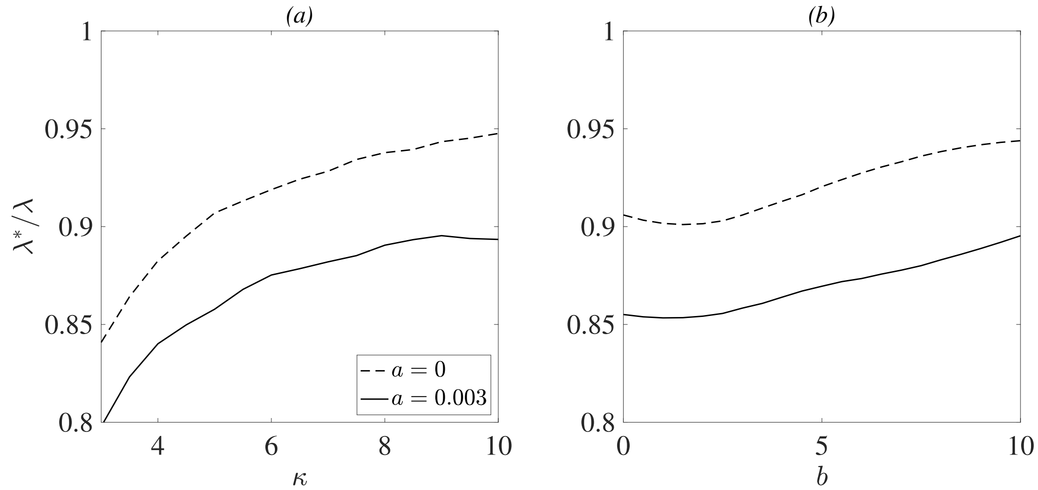

Figure 4: Diffracted wavelength of a cnoidal wave ($H = 0.25$, $\lambda = 10$) propagating over an elastic sheet ($m = 0.05$, $D = 0.1$) on a viscoelastic foundation of variable (a) stiffness κ ($b = 0$) and (b) damping coefficient b ($\kappa = 5$) with different inclination parameter a.

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