

Rotational Hypersurfaces with Constant Gauss-Kronecker Curvature

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Abstract The authors study rotational hypersurfaces with constant Gauss-Kronecker curvature in \mathbb{R}^n . They solve the ODE associated with the generating curve of such hypersurface using integral expressions and obtain several geometric properties of such hypersurfaces. In particular, they discover a class of non-compact rotational hypersurfaces with constant and negative Gauss-Kronecker curvature and finite volume, which can be seen as the higher-dimensional generalization of the pseudo-sphere.

Keywords Differential geometry, Gauss-Kronecker curvature, Ordinary differential equation

2000 MR Subject Classification 53A07

1 Introduction

In the field of differential geometry, curvature is the quantity used to measure the extent to which a geometrical object bends. In the study of submanifold geometry, the principal curvatures describe how the submanifold bends in each principal directions. The mean curvature is the mean value of all principal curvatures, while the Gauss-Kronecker curvature is the product of principal curvatures. For the rest of the paper we will call it Gauss curvature for short. Various notions of curvatures have many applications in science and industry. The problems on various restrictions on those curvatures have a long history. In particular we focus on the study of submanifolds with constant or prescribed curvature, and most of the time we only consider hypersurfaces, namely, codimension one submanifolds. The constant mean curvature (CMC for short) submanifolds can be seen as generalizations of minimal submanifolds, which are characterized as having zero mean curvature. CMC hypersurfaces enjoy good variational and geometric properties. For a detailed survey on CMC hypersurfaces in \mathbb{R}^n , we refer the readers to [1]. We mention a few results here, which motivate our work. In terms of rotational CMC surfaces in \mathbb{R}^3 , Delaunay proposed a beautiful classification theorem which indicates that the generating curves of these surfaces are formed geometrically by rolling a conic along a straight line without slippage (see [5]). In the 1980s, Hsiang and Yu generalized Delaunay's theorem

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to rotational hypersurfaces in \mathbb{R}^n (see [6–7]). Recently Buena, Galvez and Mirac studied the more general question about rotational hypersurfaces with prescribed mean curvature and obtained Delaunay-type classification theorems (see [2]).

On the other hand, higher order symmetric functions of the principal curvatures are also interesting. The r -th symmetric function of the principal curvatures is called “ r -mean curvature”, which covers the notions of mean curvature and Gauss curvature when r is equal to 1 and the dimension of the hypersurface, respectively. In 1987, Ros proved that a closed hypersurface embedded into the Euclidean space with constant r -mean curvature is a round sphere (see [9, 11]). There are also results on hypersurfaces with constant or prescribed Gauss curvature in other ambient spaces. See e.g. [12–13].

However, we notice that few examples on constant Gauss curvature hypersurfaces in \mathbb{R}^n have been constructed and studied in the literature, other than the round spheres and flat planes. According to Ros’ theorem, either such hypersurfaces are non-compact or they have non-empty boundary, if they are not the round spheres. They could still carry interesting geometric and analytic properties. The condition of having constant Gauss curvature is characterized by a Monge-Ampere type equation, and special explicit solutions to such equations can shed light on the study of general solutions. Looking for constant Gauss curvature hypersurfaces with enough symmetry could be an initial step of the study of general hypersurfaces with constant Gauss curvature. Just as Delaunay-type hypersurfaces have been built blocks of general CMC hypersurfaces in \mathbb{R}^n , hypersurfaces with special symmetry conditions can be testgrounds for general hypersurfaces with constant or prescribed Gauss curvature.

In this paper, we focus on rotational hypersurfaces M with constant Gauss curvature K in \mathbb{R}^n . Let $\gamma(t) = (\varphi(t), \psi(t))$ be a parametrization of the generating curve of M . We derive the following ODE for $\varphi(t)$:

$$K = -\frac{\varphi''(1 - \varphi'^2)^{\frac{n-3}{2}}}{\varphi^{n-2}}. \tag{1.1}$$

We solve this equation and obtain the following result.

Theorem 1.1 *Let $M \subset \mathbb{R}^n$ be a rotational hypersurface with constant Gauss curvature K such that its generating curve γ is a graph over the axis of rotation. Let $\gamma(t) = (\varphi(t), \psi(t))$ be a parametrization of the generating curve, where $\varphi(t)$ is the radius of the meridian $(n - 2)$ -sphere, $\psi(t)$ is the height function and t is the arclength parameter. Then:*

- (1) *When $K = 0$, M is a circular cone or a circular cylinder.*
- (2) *When $K \neq 0$, the expression of the inverse function of φ is locally given by*

$$t - t_0 = \int_{\varphi(t_0)}^{\varphi(t)} \pm \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}},$$

where the sign of the integrand agrees with the sign of φ' , t_0 is the initial time, C_K is a real

constant. Moreover, ψ is given by

$$\psi(t) = \psi(t_0) + \int_{t_0}^t \sqrt{1 - (\varphi')^2} dt.$$

(3) When $K < 0$ and $C_K = -1$, the corresponding hypersurface is a complete Riemannian manifold (with boundary) diffeomorphic to $S^{n-2} \times [0, +\infty)$ and has finite volume. It can be seen as a higher-dimensional generalization of the pseudosphere in dimension two.

More precise statements are made in Theorems 3.2–3.4. The pictures of the generating curves are displayed in Figures 1–2. The hypersurface in (3) in Theorem 1.1 is the only non-compact example. We also note that when $n = 3$, classifying constant curvature surfaces of revolution is a classical problem and was completely solved long ago. See e.g. [3, Chapter 3–3, Exercise 7]. Using the explicit formula for the solutions, we also study how the shape of the generating curve changes as the Gauss curvature K varies, see Subsection 3.3.

We remark that rotational hypersurfaces in space forms have been systematically studied, e.g. in [4, 8, 10]. In particular, Palmas concluded that the only complete rotational hypersurfaces (without boundary) with constant Gauss curvature in the Euclidean spaces are hyperplanes, cylinders and round spheres (see [10]). We follow the orbit geometry approach in their papers, and we allow the hypersurfaces to have boundary or to be singular. In particular, we discover a class of non-compact rotational hypersurfaces with constant and negative Gauss curvature which have finite volume. To the best of our knowledge, we have not seen such examples discussed in the literature.

2 Rotational Hypersurfaces and Its Curvatures

We set up notations and state the formulae for principal curvatures and Gauss curvature of a rotational hypersurface in \mathbb{R}^n . Let x_1, x_2, \dots, x_n denote the standard coordinates of \mathbb{R}^n and we assume that x_n is the axis of rotation. Let $f : \mathbb{R} \rightarrow (0, +\infty)$ be a smooth function.

Definition 2.1 *A hypersurface M is called a rotational hypersurface if it is produced by rotating the generating curve $x_1 = f(x_n)$ in the x_1x_n -plane around the x_n -axis. It is characterized by the following equation*

$$f(x_n)^2 = \sum_{i=1}^{n-1} x_i^2.$$

Note that $f(x_n)$ is the radius of the horizontal subsphere at height x_n . Throughout this paper, M will always denote a rotational hypersurface in \mathbb{R}^n unless otherwise stated.

We choose an appropriate parametrization of the generating curve to facilitate the calculation. Let $\varphi(t)$ denote the radius of the $n - 2$ dimensional hypersphere and $\psi(t)$ denote the corresponding height. We choose the parameter t to be the arclength parameter, that is, $\varphi'^2 + \psi'^2 = 1$. Under the above parametrization, the generating curve $x_1 = f(x_n)$ is

parametrized as $(x_1, x_n) = (\varphi(t), \psi(t))$. Note that $\varphi(t) \geq 0$ since it is the radius, and we require $\psi'(t) \geq 0$ so that the generating curve is a graph over the x_n -axis.

We use the hypersphere coordinate $(\varphi, \theta_1, \dots, \theta_{n-2})$ to parametrize the rotational hypersurface. The position vector field of rotational hypersurface M can be written as

$$\begin{aligned} \vec{r}(\varphi, \theta_1, \dots, \theta_{n-2}) \\ = (\varphi \cos \theta_1 \cdots \cos \theta_{n-2}, \varphi \cos \theta_1 \cdots \cos \theta_{n-3} \sin \theta_{n-2}, \dots, \varphi \cos \theta_1 \sin \theta_2, \varphi \sin \theta_1, \psi), \end{aligned}$$

where $\theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\theta_i \in [0, 2\pi]$ for $i = 2, 3, \dots, n - 2$. Note that ψ can be expressed in terms of φ since $\varphi'^2 + \psi'^2 = 1$.

Under the above parametrization, the principal curvatures and the Gauss curvature of M are given in the following theorems, respectively.

Theorem 2.1 *The principal curvatures k_1, \dots, k_{n-1} of M are given below:*

- (1) $k_1 = -\frac{\varphi''}{\psi'}$;
- (2) $k_i = \frac{\psi'}{\varphi}$ for $i = 2, 3, \dots, n - 1$.

Theorem 2.2 *The Gauss curvature K of M is given below:*

$$K = -\frac{\varphi''\psi'^{n-3}}{\varphi^{n-2}} (n \geq 3).$$

Detailed calculations can be found in [2, Section 2].

3 Analysis of Rotational Hypersurface with Constant Gauss Curvature

We require the Gauss curvature of rotational hypersurface M to be a constant K . Then the equation in Theorem 2.2 is transformed into an ODE as below:

$$K = -\frac{\varphi''\psi'^{n-3}}{\varphi^{n-2}} = -\frac{\varphi''(1 - \varphi'^2)^{\frac{n-3}{2}}}{\varphi^{n-2}}. \tag{3.1}$$

This equation will be the main equation that we study in this paper. Here we require that $\psi' \geq 0$, so that the generating curve is a graph over the x_n -axis. In this section, we will solve this equation by separation of variables.

3.1 Solutions to the ODE

When $K = 0$, we get

$$\varphi''(1 - \varphi'^2)^{\frac{n-3}{2}} = 0.$$

Obviously, we must have $\varphi'' \equiv 0$ or $\varphi' \equiv \pm 1$. Both yields

$$\varphi(t) = c_1 t + c_2. \tag{3.2}$$

Thus we have the following theorem.

Theorem 3.1 *A rotational hypersurface with constant Gauss curvature $K = 0$ is one of the following:*

- (1) *A right straight cylinder in \mathbb{R}^n .*
- (2) *A right circular cone in \mathbb{R}^n .*

Proof From (3.2), we know that the generating curve is a straight line in the case where $K = 0$. Consider the equation

$$\varphi(t) = c_1 t + c_2,$$

when $c_1 = 0$, φ is a constant in which case M is a right straight cylinder. Otherwise, when $c_1 \neq 0$, M is a right circular cone.

Remark 3.1 In fact, the Gauss curvature of any cylinder or cone is 0.

In the rest of the paper, we will therefore discuss the case where $K \neq 0$. We rewrite (3.1) in the following form:

$$K\varphi^{n-2} = -\varphi''(1 - \varphi'^2)^{\frac{n-3}{2}}. \tag{3.3}$$

Multiply both sides by φ' and integrate both sides, we have

$$K\varphi^{n-1} = (1 - \varphi'^2)^{\frac{n-1}{2}} + C_K, \tag{3.4}$$

where C_K is a constant to be chosen.

Since φ is the radius parameter, we only consider the case where $\varphi \geq 0$. First, we notice that the solution φ is bounded.

Lemma 3.1 *φ is a bounded function such that*

- (1) *for $K > 0$, $\max\{0, \frac{C_K}{K}\} \leq \varphi^{n-1} \leq \frac{C_K+1}{K}$ where $C_K > -1$;*
- (2) *for $K < 0$, $\max\{0, \frac{C_K+1}{K}\} \leq \varphi^{n-1} \leq \frac{C_K}{K}$ where $C_K < 0$.*

Proof From (3.4), we get

$$K\varphi^{n-1} - C_K = (1 - \varphi'^2)^{\frac{n-1}{2}}.$$

Clearly, we know that $0 \leq (1 - \varphi'^2)^{\frac{n-1}{2}} \leq 1$. So, we have

$$C_K \leq K\varphi^{n-1} \leq C_K + 1.$$

For $K > 0$, we further yield

$$\frac{C_K}{K} \leq \varphi^{n-1} \leq \frac{C_K + 1}{K}.$$

Since we only consider the case where $\varphi \geq 0$, we have

$$\max\left\{0, \frac{C_K}{K}\right\} \leq \varphi^{n-1} \leq \frac{C_K + 1}{K}.$$

Here, we must have $C_K > -1$ to make sure that $\frac{C_K+1}{K} > 0$.

Similarly, we can deduce the inequality for $K < 0$,

$$\max \left\{ 0, \frac{C_K + 1}{K} \right\} \leq \varphi^{n-1} \leq \frac{C_K}{K}.$$

Here, we must have $C_K < 0$ to make sure that $\frac{C_K}{K} > 0$.

Now we solve the ODE when $K > 0$ and $K < 0$, respectively.

Theorem 3.2 *Suppose $K > 0$. Let φ be a solution to the ODE (3.3), then:*

(1) *The inverse function of φ is given by*

$$t - t_0 = \int_{\varphi(t_0)}^{\varphi(t)} \pm \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}},$$

where t_0 is a fixed initial time.

(2) *The solution φ can be defined on the interval $I = [C', C' + T]$, where C' is a real number,*

$$T = T(C_K) = 2 \int_{(\max\{0, \frac{C_K}{K}\})^{\frac{1}{n-1}}}^{(\frac{C_K+1}{K})^{\frac{1}{n-1}}} \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}}$$

and $\varphi(C') = \varphi(C' + T) = \max \left\{ 0, \frac{C_K}{K} \right\}^{\frac{1}{n-1}}$.

(3) *The sign of the integrand is + for $t \in [C', C' + \frac{T}{2}]$ and - for $t \in [C' + \frac{T}{2}, C' + T]$, or the other way around if the orientation of the generating curve is reversed.*

Proof From (3.4), we get

$$\varphi' = \pm \sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}} \tag{3.5}$$

and

$$dt = \pm \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}}.$$

Here, the sign of dt agrees with the sign of φ' .

Then integrate both sides of the above equation, and we get

$$t - t_0 = \int_{\varphi(t_0)}^{\varphi(t)} \pm \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}}. \tag{3.6}$$

We also note that the solution φ is invariant under time translation and reversion, and thus the value of t_0 does not affect the shape of the generating curve.

Now, we should consider the interval of definition for this solution. From Lemma 3.1, we know that the integrand in (3.6) is bounded from both above and below. We try to integrate

from the lower bound to the upper bound and show that the integral converges, that is, we claim

$$T' = \int_{\left(\max\left\{0, \frac{C_K}{K}\right\}\right)^{\frac{1}{n-1}}}^{\left(\frac{C_K+1}{K}\right)^{\frac{1}{n-1}}} \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}} < +\infty. \tag{3.7}$$

To prove the claim, we only need to check the singularity when φ reaches $\left(\frac{C_K+1}{K}\right)^{\frac{1}{n-1}}$. Let $A = \left(\frac{C_K+1}{K}\right)^{\frac{1}{n-1}} \neq 0$, and the Taylor expansion of the integrand as $A - \varphi \rightarrow 0$ is given below:

$$T' = \int_{\left(\max\left\{0, \frac{C_K}{K}\right\}\right)^{\frac{1}{n-1}}}^A \frac{1}{\sqrt{2A^{n-2}K}} (A - \varphi)^{-\frac{1}{2}} + O((A - \varphi)^{-1}) d\varphi.$$

Since the order of the main term of the integrand in terms of $(\varphi - A)$ is greater than -1 , the claim is clearly true.

Thus the solution $\varphi(t)$ can be defined on a time interval of length T' , say $[t_0, t_0 + T']$. Without loss of generality, we may assume that $\varphi(t)$ is increasing on $[t_0, t_0 + T']$, that is, the sign of the integrand in (3.6) is positive. In this way $\varphi(t)$ reaches its minimum at $t = t_0$ and its maximum at $t = t_0 + T'$.

Now we extend the solution $\varphi(t)$ to the interval $[t_0, t_0 + 2T']$ by reflection. Namely, we define $\varphi(t) = \varphi(2t_0 + 2T' - t)$. Since (3.3) is invariant under time translation and reversion, this extension of φ is a solution to the equation. By checking that the $(2n + 1)^{st}$ order derivatives of φ at $t = t_0 + T'$ equal zero, we know that the left derivatives and right derivatives of φ agree at $t = t_0 + T'$. Therefore, we know that $\varphi(t)$ is smooth for $t \in [t_0, t_0 + 2T']$.

Thus we obtain a solution φ on $[t_0, t_0 + T]$ satisfying all the desired properties, where

$$T = 2T' = 2 \int_{\left(\max\left\{0, \frac{C_K}{K}\right\}\right)^{\frac{1}{n-1}}}^{\left(\frac{C_K+1}{K}\right)^{\frac{1}{n-1}}} \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}}. \tag{3.8}$$

Remark 3.2 When $C_K = 0$, the solution becomes $\varphi(t) = \frac{\cos(\sqrt{K}t - \theta_0)}{\sqrt{K}}$. In this case, M is the round sphere of constant Gauss curvature K .

Recall that $\varphi'^2 + \psi'^2 = 1$, by (3.4) we have

$$\psi(t) = \psi(t_0) + \int_{t_0}^t \sqrt{1 - \varphi'(s)^2} ds = \psi(t_0) + \int_{t_0}^t (K\varphi(s)^{n-1} - C_K)^{\frac{1}{n-1}} ds.$$

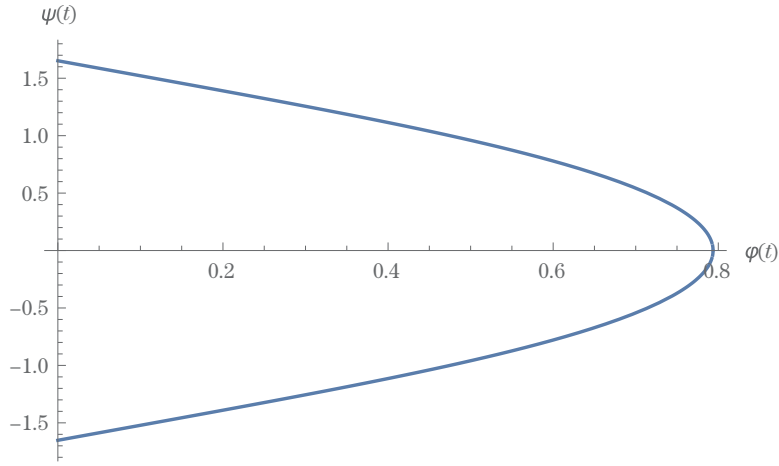
Thus using the parametrization $(\varphi(t), \psi(t))$ of the generating curve, we draw the pictures of the generating curves using Mathematica. Figures 1(a) and 1(b) show the generating curves for $K = 1, C_K = -0.5$ and $K = 1, C_K = 2$, respectively.

Similarly, we can describe the solution for $K < 0$ as follows.

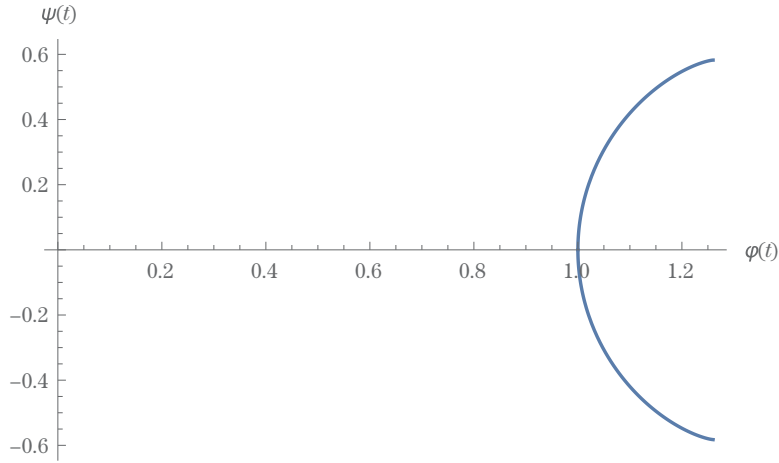
Theorem 3.3 *Suppose $K < 0$. Let φ be a solution to the ODE (3.3), then:*

(1) *The inverse function of φ is given by*

$$t - t_0 = \int_{\varphi(t_0)}^{\varphi(t)} \pm \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}},$$



(a) $K = 1, C_K = -0.5$



(b) $K = 1, C_K = 2$

Figure 1

where t_0 is a fixed initial time.

(2) When $C_K \neq -1$, φ can be defined on $[C', C' + T] \cup [D' - T, D']$, where C', D' are two real numbers such that $\varphi(C') = \varphi(D') = (\frac{C_K}{K})^{\frac{1}{n-1}}$ where

$$D' - C' = 2 \int_{(\frac{C_K+1}{K})^{\frac{1}{n-1}}}^{(\frac{C_K}{K})^{\frac{1}{n-1}}} \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}}$$

and

$$T = T(C_K) = \int_{(\max\{0, \frac{C_K+1}{K}\})^{\frac{1}{n-1}}}^{(\frac{C_K}{K})^{\frac{1}{n-1}}} \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}}.$$

In this case, the sign of the integrand is $-$ in the interval $[C', C' + T]$ and $+$ in the interval $[D' - T, D']$. Or the other way around if the orientation of the generating curve is reversed.

(3) When $C_K = -1$, we fix the sign of the integrand to be positive. Under this convention, the interval of definition of the solution to (3.3) extends to $-\infty$. In particular, the corresponding hypersurface is non-compact complete and unbounded in the x_n -direction.

Proof Similar to Theorem 3.2, we can derive the inverse function of φ as follows

$$t - t_0 = \int_{\varphi(t_0)}^{\varphi(t)} \pm \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}}.$$

When $C_K \neq -1$, we also claim

$$T = \int_{\left(\frac{C_K+1}{K}\right)^{\frac{1}{n-1}}}^{\left(\frac{C_K}{K}\right)^{\frac{1}{n-1}}} \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}} < +\infty.$$

To prove the claim, we similarly let $A = \left(\frac{C_K+1}{K}\right)^{\frac{1}{n-1}} \neq 0$. Then, the Taylor expansion of the integrand as $\varphi - A \rightarrow 0$ is given by

$$T = \int_A^{\left(\frac{C_K}{K}\right)^{\frac{1}{n-1}}} \frac{1}{\sqrt{-2A^{n-2}K}} (\varphi - A)^{-\frac{1}{2}} + O((\varphi - A)^{-1}) d\varphi.$$

Clearly, the integral converges since the order of the integrand's main term is greater than -1 .

Thus by the same reflection argument as in the proof of Theorem 3.2, we can show that φ can be extended to a smooth solution defined on $[t_0, t_0 + 2T]$, where T is the above integral and t_0 can be any real number. Moreover, we can prescribe its monotonicity by fixing the orientation of the generating curve. However, in this way φ can be negative somewhere. After deleting the interval on which φ is negative, we obtain the desired form of the domain of definition of φ .

Finally, when $C_K = -1$, we need to prove that the integral in (3.3) diverges, namely

$$T = \int_0^{\left(\frac{-1}{K}\right)^{\frac{1}{n-1}}} \frac{d\varphi}{\sqrt{1 - (K\varphi^{n-1} + 1)^{\frac{2}{n-1}}}} = +\infty.$$

Thus the interval of definition of φ can extend to $-\infty$.

We consider the behavior of the integrand when

$$\varphi \rightarrow \left(\frac{C_K + 1}{K}\right)^{\frac{1}{n-1}} = 0.$$

Let $x = K\varphi^{n-1}$, where $x < 0$. Then let

$$f(x) = 1 - (x + 1)^{\frac{2}{n-1}}.$$

After expanding $f(x)$ at $x = 0$ with Taylor Series, we get

$$f(x) = 1 - \left(1 + \frac{2}{n-1}x + O(x^2)\right) = -\frac{2}{n-1}x + O(x^2).$$

Then we can rewrite the integrand as

$$\frac{1}{\sqrt{1 - (K\varphi^{n-1} - C_K)^{\frac{2}{n-1}}}} = \frac{1}{\sqrt{f(x)}} = \frac{1}{\sqrt{-\frac{2}{n-1}x + O(x^2)}}. \tag{3.9}$$

Clearly, we know that the order of the integrand in terms of φ is equal to the order of its main term, namely $\frac{1-n}{2}$ because $x = K\varphi^{n-1}$. When $n > 3$, we know that the order of the integrand is less than -1 , which implies that the integral diverges when $\varphi \rightarrow 0$.

Now we see that as t goes to $-\infty$, $\varphi \rightarrow 0$ and $\varphi' = \sqrt{1 - (K\varphi^{n-1} + 1)^{\frac{2}{n-1}}} \rightarrow 0$ (we have fixed the sign of φ' to be positive). Thus $\psi' = \sqrt{1 - (\varphi')^2} \rightarrow 1$ and $\psi \rightarrow -\infty$ (because we let t decrease to $-\infty$). Recall that ψ is the height function. This shows that the generating curve is an unbounded curve asymptotic to the axis of rotation. Thus the rotational hypersurface corresponding to $C_K = -1$ is complete as a metric space.

Remark 3.3 The hypersurface corresponding to $C_K = -1$ in the above theorem can be seen as a higher-dimensional generalization of the pseudo-sphere in dimension two. Our results do not contradict Ros' theorem in [11] since the hypersurfaces in our theorem have non-empty boundary. We also note that we can find a non-compact solution only when $K < 0$ and $C_K = -1$. In all other cases, the corresponding hypersurfaces are compact with boundary or compact with conical singularities at the axis of rotation.

Using Mathematica, we draw the generating curves for $K < 0$. Figure 2(a) depicts the generating curve of the non-compact hypersurface corresponding to $C_K = -1$, while Figures 2(b) and 2(c) show the generating curves for $K = -1$, $C_K = -0.5$ and $K = -1$, $C_K = -2$, respectively.

Using the integral expression of the solution φ , we can calculate its Taylor expansions at critical points in order to extract more information about the local behavior of φ .

Lemma 3.2 *The series expansion of $\varphi(t)$ near $\varphi(t_0) = \varphi_{\max}$ when $K > 0$ is given by*

$$\begin{aligned} \varphi(t) = & \left(\frac{C_K + 1}{K}\right)^{\frac{1}{n-1}} \left\{ 1 - \frac{K}{2} \left(\frac{C_K + 1}{K}\right)^{\frac{n-3}{n-1}} (t - t_0)^2 \right. \\ & \left. - \frac{K^2}{24} \left(\frac{C_K + 1}{K}\right)^{\frac{2(n-3)}{n-1}} \left[\frac{(n-3)(C_K + 1)}{K} - (n-2) \right] (t - t_0)^4 - \dots \right\}. \end{aligned} \tag{3.10}$$

Proof By (3.5), we know that the first order derivative of φ near its maximum is zero. Then, we can further compute its second order derivative as

$$\varphi''(t_0) = -(K\varphi^{n-1} - C_K)^{\frac{3-n}{n-1}} \cdot K\varphi^{n-2} = -K \left(\frac{C_K + 1}{K}\right)^{\frac{n-2}{n-1}}.$$

Similarly, we can compute its 4th order derivative, and so on. Notice that the sign of these derivatives are negative near φ_{\max} , we can get the series shown above by using Taylor expansion.

Lemma 3.3 *The series expansion of $\varphi(t)$ near $\varphi(t_0) = \varphi_{\min}$ when $K < 0$ and $C_K < -1$ is*

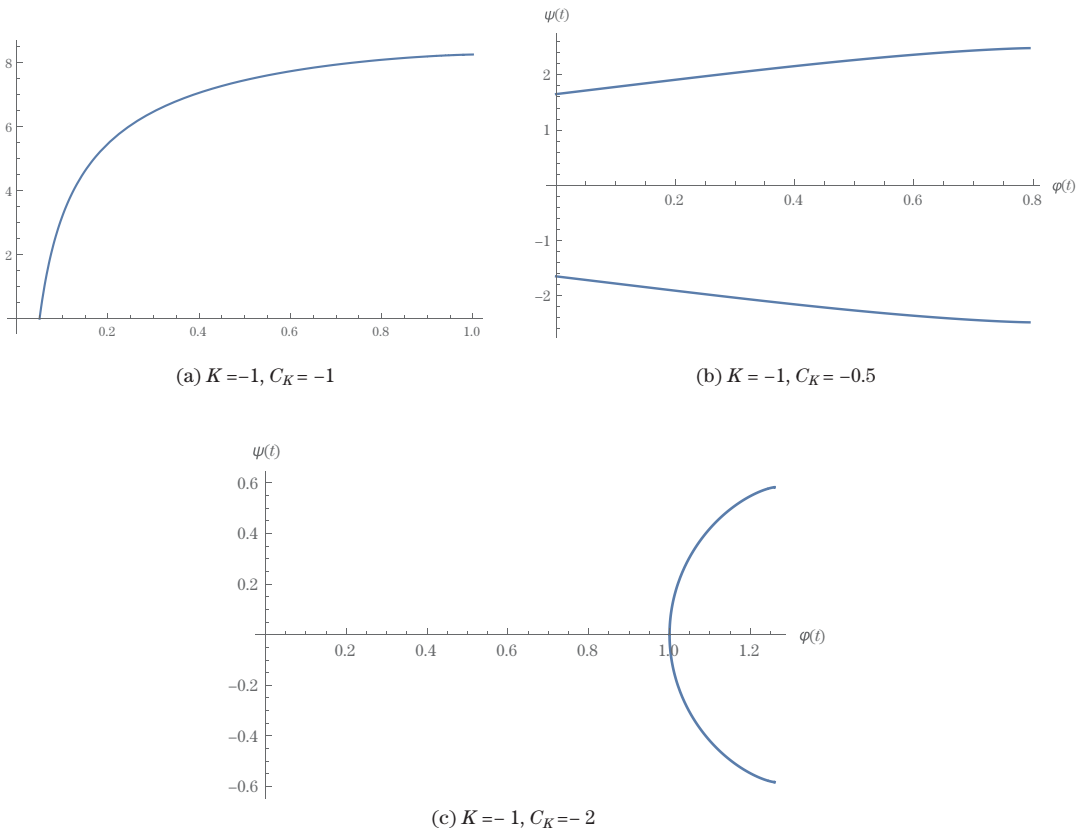


Figure 2

given by

$$\begin{aligned} \varphi(t) = & \left(\frac{C_K + 1}{K}\right)^{\frac{1}{n-1}} \left\{ 1 + \frac{K}{2} \left(\frac{C_K + 1}{K}\right)^{\frac{n-3}{n-1}} (t - t_0)^2 \right. \\ & \left. + \frac{K^2}{24} \left(\frac{C_K + 1}{K}\right)^{\frac{2(n-3)}{n-1}} \left[\frac{(n-3)(C_K + 1)}{K} - (n-2) \right] (t - t_0)^4 + \dots \right\}. \end{aligned} \quad (3.11)$$

Proof Similar to Lemma 3.2, we can compute the $2k^{\text{th}}$ order derivatives near the minimum $\varphi(t_0) = \left(\frac{C_K + 1}{K}\right)^{\frac{1}{n-1}}$.

For the non-compact hypersurface, we also have the asymptotic expansion of φ near infinity.

Lemma 3.4 *Up to time translation, the asymptotic expansion of $\varphi(t)$ for $K < 0$ and $C_K = -1$ in Theorem 3.3 near $-\infty$ is given by*

$$\varphi(t) = f(n)|t|^{\frac{2}{3-n}} + g(n)|t|^{\frac{2n}{3-n}} + O(|t|^{\frac{4n-2}{3-n}}).$$

Here, f, g are given by

$$f(n) = \left(\frac{1}{AB}\right)^{\frac{2}{3-n}}$$

and

$$g(n) = \frac{C}{B} \cdot \frac{2}{3-n} \left(\frac{1}{AB}\right)^{\frac{2n}{3-n}},$$

where $A = \sqrt{\frac{n-1}{2}}$, $B = \frac{2}{3-n}|K|^{-\frac{1}{2}}$ and $C = \frac{n-3}{2(n^2-1)}|K|^{\frac{1}{2}}$.

Proof By expanding the integrand in Theorem 3.3, we get

$$\begin{aligned} t &= \sqrt{\frac{n-1}{2}} \int |K|^{-\frac{1}{2}} \varphi^{\frac{1-n}{2}} + |K|^{\frac{1}{2}} \frac{n-3}{4(n-1)} \varphi^{\frac{n-1}{2}} + O(\varphi^{\frac{3n-3}{2}}) d\varphi \\ &= \sqrt{\frac{n-1}{2}} \cdot \left[\frac{2}{3-n} |K|^{-\frac{1}{2}} \varphi^{\frac{3-n}{2}} + \frac{n-3}{2(n^2-1)} |K|^{\frac{1}{2}} \varphi^{\frac{n+1}{2}} + O(\varphi^{\frac{3n-1}{2}}) \right]. \end{aligned}$$

Note that the integral in the first line gives us an undetermined constant. Up to a translation of t , we can take that constant to be 0. Let $A = \sqrt{\frac{n-1}{2}}$, $B = \frac{2}{3-n}|K|^{-\frac{1}{2}}$ and $C = \frac{n-3}{2(n^2-1)}|K|^{\frac{1}{2}}$, we can compute that

$$\varphi(t) = f(n)|t|^{\frac{2}{3-n}} + g(n)|t|^{\frac{2n}{3-n}} + O(|t|^{\frac{4n-2}{3-n}}).$$

3.2 Finite volume of the noncompact hypersurfaces

For the hypersurface described in Theorem 3.3 when $C_K = -1$, we will show that its “surface area” and “volume” of the region enclosed by the hypersurface are indeed finite.

Before the proof, we introduce the following notations. Let $V_n(r)$ denote the volume of an n -dimensional ball of radius r , and $S_n(r)$ denote the area of an n -dimensional sphere of radius r . It is well-known that

$$V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \cdot r^n, \quad S_n(r) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \cdot r^{n-1}.$$

Theorem 3.4 *The surface area of the hypersurface in Theorem 3.3 when $C_K = -1$ is finite. Moreover, the volume of the region enclosed by the hypersurface and the horizontal disk at the end of the hypersurface is also finite.*

Proof The surface area of the rotational hypersurface is

$$S = 2 \int_{-\infty}^{t_0} S_{n-1}(\varphi) d\psi = 2 \int_{-\infty}^{t_0} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \cdot \varphi^{n-2} d\psi.$$

Here, $t = t_0$ is the point where $\varphi(t)$ reaches its maximum. Without loss of generality, we can take $t_0 = 0$ since $\varphi(t)$ is invariant under translation, namely

$$S = 2 \int_{-\infty}^0 \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \cdot \varphi^{n-2} d\psi.$$

From Theorem 3.3, we know that $\varphi' \rightarrow 0$ as $t \rightarrow -\infty$, which also implies $\psi' \rightarrow 1$ since $\varphi'^2 + \psi'^2 = 1$.

So, we know that as $t \rightarrow -\infty$,

$$\frac{d\psi}{dt} \rightarrow 1.$$

Consider

$$S(t) = 2 \int_t^0 \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \cdot \varphi^{n-2} d\psi.$$

Let $\text{ord}(S)$ be the order of $S(t)$ in terms of $|t|$, namely $S(t) = O(|t|^{\text{ord}(S)})$.

From Theorem 3.3, we know that the order of the integral in (3.6) is $\frac{1-n}{2}$, which indicates that the order of $|t|$ in terms of φ is $\frac{3-n}{2}$. Therefore, the order of φ in terms of $|t|$ is $\frac{2}{3-n}$.

Then, we get

$$\text{ord}(S) = \frac{2}{3-n} \cdot (n-2) + 1 = \frac{n-1}{3-n}. \tag{3.12}$$

Since $n > 3$, we know that

$$\frac{n-1}{3-n} = -1 + \frac{2}{3-n} < -1.$$

Therefore, the surface area of this non-compact hypersurface is finite.

Similarly, we can derive the expression of the volume of the enclosed region

$$V = 2 \int_{-\infty}^0 V_{n-1}(\varphi) dt = 2 \int_{-\infty}^0 \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2} + 1)} \cdot \varphi^{n-1} dt.$$

Consider

$$V(t) = 2 \int_t^0 \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2} + 1)} \cdot \varphi^{n-1} dt.$$

Then we get

$$\text{ord}(V) = \frac{2}{3-n} \cdot (n-1) + 1 = \frac{n+1}{3-n}. \tag{3.13}$$

When $n > 3$, we know that

$$\frac{n+1}{3-n} = -1 + \frac{4}{3-n} < -1.$$

Clearly, it indicates that the integral converges as $t \rightarrow -\infty$, and thus the volume is also finite.

Remark 3.4 We also compute the approximate value of the volume of the enclosed region and the surface area of this hypersurface when $n = 4$ and $K = 1$ by Mathematica, which are 1.82 and 19.74, respectively.

3.3 A comparison theorem

From (3.6), we know that the value of $\varphi(t)$ for a given value of t is dependent on the Gauss curvature K . When the Gauss curvature is a constant K , let φ_K denote the solution to (3.6) described in Theorems 3.2 or 3.3 and ψ_K denote the corresponding height function. We would like to study the behavior of the solution φ_K when K changes.

In the following theorem, we show that for $K > 0$, the value of φ_K at a fixed height $\psi_K = y$ decreases as K increases if the maximum of φ_K is fixed. Geometrically the generating curve drops faster to the axis of rotation for greater positive Gauss curvature.

Theorem 3.5 *Take $a, b \in \mathbb{R}$ and $a > b > 0$. Assume that both φ_a and φ_b obtain the same maximum at $t = t_0$, namely $\varphi_a(t_0) = \varphi_b(t_0) = \varphi_{\max} = C$. We also assume that on a small interval $D = [t_0, t_0 + \delta]$, both φ_a and φ_b are monotonically decreasing, and ψ_a and ψ_b are increasing. Then $\forall y \in \psi_a(D) \cup \psi_b(D)$, we get*

$$\varphi_a(\psi_a^{-1}(y)) \leq \varphi_b(\psi_b^{-1}(y)).$$

Proof From Theorem 3.1, we get

$$C^{n-1} = \varphi_{\max}^{n-1} = \frac{C_K + 1}{K}.$$

Recall that $\varphi'^2 + \psi'^2 = 1$ and the expression of φ' in (3.5), we get

$$\begin{aligned} \psi_K(t) &= \int_{t_0}^t \sqrt{1 - \varphi_K'^2} dt \\ &= \int_{\varphi_K(t_0)}^{\varphi_K(t)} \frac{\sqrt{1 - (1 - (K\varphi_K^{n-1} - C_K)^{\frac{2}{n-1}})}}{\sqrt{1 - (K\varphi_K^{n-1} - C_K)^{\frac{2}{n-1}}}} d\varphi_K \\ &= \int_{\varphi_K(t_0)}^{\varphi_K(t)} \frac{1}{\sqrt{\frac{1}{(K\varphi_K^{n-1} - C_K)^{\frac{2}{n-1}}}}} d\varphi_K \\ &= \int_{\varphi_K(t_0)}^{\varphi_K(t)} \frac{1}{\sqrt{\frac{1}{(K\varphi_K^{n-1} - (C^{n-1}K - 1))^{\frac{2}{n-1}}}}} d\varphi_K \\ &= \int_{\varphi_K(t_0)}^{\varphi_K(t)} \sqrt{K\varphi_K^{n-1} - (C^{n-1}K - 1)^{\frac{2}{n-1}}} d\varphi_K. \end{aligned}$$

Thus, we have

$$y = \psi_K(\psi_K^{-1}(y)) = \int_{\varphi_K(t_0)}^{\varphi_K(\psi_K^{-1}(y))} \sqrt{K\varphi_K^{n-1} - (C^{n-1}K - 1)^{\frac{2}{n-1}}} d\varphi_K. \tag{3.14}$$

Obviously, we know that the integrand $f(K) = \sqrt{K\varphi_K^{n-1} - (C^{n-1}K - 1)^{\frac{2}{n-1}}}$ increases as $|K|$ increases.

Therefore, for a fixed maximum $\varphi_a(t_0) = \varphi_b(t_0) = C$ and negative φ' , we must have $\varphi_a(\psi_a^{-1}(y)) \leq \varphi_b(\psi_b^{-1}(y))$ to make sure that the left side of (3.14) remains the same.

Similarly, we propose a parallel theorem for $K < 0$. In this case, the generating curve stays further away from the axis of rotation when $|K|$ increases.

Theorem 3.6 *Take $a, b \in \mathbb{R}$ and $0 > a > b$. Assume that both φ_a and φ_b obtain the same minimum at $t = t_0$, namely $\varphi_a(t_0) = \varphi_b(t_0) = \varphi_{\min} = C$. We also assume that on a small interval $D = [t_0, t_0 + \delta]$, both φ_a and φ_b are monotonically increasing, and ψ_a and ψ_b are increasing. Then $\forall y \in \psi_a(D) \cup \psi_b(D)$, we get*

$$\varphi_a(\psi_a^{-1}(y)) \leq \varphi_b(\psi_b^{-1}(y)).$$

Proof First consider the case where $C = \varphi_{\min} \neq 0$.

Similarly to Theorem 3.5, we get

$$\begin{aligned} \psi_K(t) &= \int_{t_0}^t \sqrt{1 - \varphi_K'^2} dt \\ &= \int_{\varphi_K(t_0)}^{\varphi_K(t)} \sqrt{K\varphi_K^{n-1} - (C^{n-1}K - 1)^{\frac{2}{n-1}}} d\varphi_K. \end{aligned}$$

Thus, we have

$$y = \psi_K(\psi_K^{-1}(y)) = \int_{\varphi_K(t_0)}^{\varphi_K(\psi_K^{-1}(y))} \sqrt{K\varphi_K^{n-1} - (C^{n-1}K - 1)^{\frac{2}{n-1}}} d\varphi_K. \tag{3.15}$$

From Theorem 3.5, we know that the integrand $\sqrt{K\varphi_K^{n-1} - (C^{n-1}K - 1)^{\frac{2}{n-1}}}$ increases as $|K|$ increase.

Therefore, for a fixed minimum $\varphi_a(t_0) = \varphi_b(t_0) = C$ and positive φ' , we must have $\varphi_a(\psi_a^{-1}(y)) \leq \varphi_b(\psi_b^{-1}(y))$ to make sure that the left side of (3.15) remains the same.

When $C = \varphi_{\min} = 0$, then C_K will be a constant that is independent of K . In this case, the proof is exactly the same.

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