ON CLOSED SIX-MANIFOLDS ADMITTING METRICS WITH POSITIVE SECTIONAL CURVATURE AND NON-ABELIAN SYMMETRY

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Abstract. We study the topology of closed, simply-connected, 6-dimensional Riemannian manifolds of positive sectional curvature which admit isometric actions by $SU(2)$ or $SO(3)$. We show that their Euler characteristic agrees with that of the known examples, i.e. S^6, $\mathbb{CP}^3$, the Wallach space $SU(3)/T^2$ and the biquotient $SU(3)//T^2$. We also classify, up to equivariant diffeomorphism, certain actions without exceptional orbits and show that there are strong restrictions on the exceptional strata.

1. Introduction

The study of Riemannian manifolds with positive sectional curvature is an old and fundamental subject in Riemannian geometry. There are very few compact examples of positively curved manifolds besides the so-called Compact Rank One Symmetric Spaces, which we will abbreviate as CROSS. In fact, the only further known examples occur only in dimension less than or equal to 24 and consist of homogeneous spaces [Wal72], [BB76], biquotients [Esc82], [Baz96] and one cohomogeneity one manifold in dimension 7 [GVZ11] and [Dea11].

The fundamental group of a compact Riemannian manifold with positive sectional curvature is finite, and it is trivial or equal to $\mathbb{Z}/2\mathbb{Z}$ in even dimensions. Furthermore, odd-dimensional positively curved closed manifolds are orientable. However, for simply connected closed manifolds, no general topological obstructions are known to separate the class of positively curved manifolds from the class of non-negatively curved manifolds, although there are many examples known to admit non-negative curvature.

There are several classification results for positively curved manifolds in low dimensions, though all of which require some "symmetry" conditions on the metric. Positively curved 3-manifolds are space forms [Ham82]. In dimension 4, Hsiang and Kleiner showed that positively curved simply connected 4-manifolds with S^1 symmetry are homeomorphic to the 4-sphere S^4 or the projective space $\mathbb{CP}^2$ in [HK89]; later Grove and Wilking improved the result to equivariant diffeomorphism in [GW14]. In dimension 5, Xiaochun Rong showed that a T^2-invariant simply connected closed 5-manifold is homeomorphic to a 5-sphere in [Ron02]. By the work of Barden [Bar65] and Smale [Sma62], we know that there are no exotic 5-spheres, and thus such a 5-manifold is actually diffeomorphic to the standard 5-sphere.

Inspired by Hsiang and Kleiner's work, Karsten Grove proposed what is now called the "symmetry program" in [Gro02], which is to study positively curved manifolds with "large" symmetry group. Here "large" can have several different meanings. Many results were obtained in this direction, particularly for torus actions. For example, Grove and Searle proved the Maximal Rank theorem in [GS94], which

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states that the symmetry rank of an n-dimensional positively curved closed manifold is at most $[\frac{n+1}{2}]$, and in the case of equality the manifold is diffeomorphic to a sphere, $\mathbb{RP}^n$, $\mathbb{CP}^n$ or a lens space. In [FR05], Fuquan Fang and Xiaochun Rong showed that a closed simply connected $2m$-manifold ($m \geq 5$) of positive sectional curvature on which an $(m-1)$-torus acts isometrically is homeomorphic to a complex projective space if and only if its Euler characteristic is not 2. Burkhard Wilking showed that when $n \geq 10$ and a positively curved closed simply connected n-manifold M has symmetry rank at least $\frac{n}{4}+1$, M is homeomorphic to S^n or $\mathbb{HP}^{\frac{n}{4}}$ or homotopy equivalent to $\mathbb{CP}^{\frac{n}{2}}$ in [Wil03]. Recently Kennard, Wiemeler and Wilking claimed that an even-dimensional positively curved manifold with T^5-symmetry has Euler characteristic at least 2 [Ken20].

After these classification results of positively curved manifolds invariant under torus actions, it is natural to investigate metrics with non-abelian symmetry. Wilking studied positively curved manifolds with high symmetry degree or low cohomogeneity relative to the dimension in [Wil06]. Since all non-abelian compact Lie groups contain a rank 1 subgroup, $SU(2)$ or $SO(3)$, it is natural to study metrics invariant under $SU(2)$ or $SO(3)$. In dimension 5, Fabio Simas obtained a partial classification of positively curved 5-manifolds invariant under $SU(2)$ or $SO(3)$ in [Sim16]. We point out here that Fabio Simas listed $SU(3)/SO(3)$ with the linear $SU(2)$-action as a candidate for positive curvature. But we note that this is not possible, since the fixed point set $(SU(3)/SO(3))^{\mathbb{Z}/2\mathbb{Z}}$ is diffeomorphic to $U(2)/O(2)$ which does not admit positive curvature.

In this paper we study 6-dimensional positively curved manifolds with $SU(2)$ or $SO(3)$ symmetry. This is also the first dimension where new examples other than the CROSSes have been constructed, which need to be recognized. They are the Wallach space $SU(3)/T^2$, where T^2 is the maximal torus, and the biquotient $SU(3)//T^2$, where

$$T^2 = \{(\mathrm{diag}(z, w, zw),\ \mathrm{diag}(1, 1, \bar{z}^2\bar{w}^2)) | z, w \in S^1\} \subset S(U(3) \times U(3))$$

acts freely on $SU(3)$. On the first space one has an action by both $SO(3)$ and $SU(2)$, isometric in the positively curved metric, and on the second space an action by $SU(2)$ which commutes with $\mathrm{diag}(1, 1, \bar{z}^2\bar{w}^2)$.

Our first result is:

Theorem 1.1. *Let $M = M^6$ be a 6-dimensional closed simply connected Riemannian manifold of positive sectional curvature such that $SU(2)$ or $SO(3)$ acts isometrically and effectively on M. Then:*

- *(a) The Euler characteristic $\chi(M) = 2,\ 4,\ 6$;*
- *(b) The principal isotropy group is trivial unless M is equivariantly diffeomorphic to S^6 with a linear $SO(3)$-action;*
- *(c) When the principal isotropy is trivial, the exceptional isotropy groups are either cyclic or dihedral groups.*

Notice that in the known examples, one has indeed $\chi(M) = 2,\ 4,\ 6$. Before stating the next theorems, we mention that the orbit space M/G is homeomorphic to a 3-sphere or a 3-ball (see Theorem 3.1) unless M is equivariantly diffeomorphic to S^6 with a linear $SO(3)$-action (See Section 5, Example 1(b)).

In the case of $G = SU(2)$ we will show:

Theorem 1.2. *Let M be a positively curved simply connected 6-manifold. Assume that $G = SU(2)$ acts on M isometrically and effectively.*

- *(a) If the fixed point set M^G is non-empty, then M is equivariantly diffeomorphic to a linear action on S^6 or $\mathbb{CP}^3$.*

(b) *If M^G is empty and the action has no exceptional orbits, then M is diffeomorphic to S^6, $S^2 \times S^4$ or $SU(3)/T^2$.*

In the case of $G = SO(3)$ we have:

Theorem 1.3. *Let M be a positively curved simply connected 6-manifold. Assume that $G = SO(3)$ acts on M isometrically and effectively and that the orbit space M/G is a 3-ball whose boundary contains more than 1 orbit type, and that there are no exceptional orbits or interior singular orbits. Then M^6 is equivariantly homeomorphic to a linear action on S^6.*

See Theorem 4.2 for further results in this special case. Explicit actions as in the above theorems are described in Section 5.

The strategy to obtain these results is to analyze the structure of the orbit space and recover M from M/G. We will show that M/G is homeomorphic to B^4, B^3 or S^3. When $M/G = B^4$, the action is fixed point homogeneous and is completely understood. When $M/G = B^3$, we characterize the boundary and interior stratification for both $G = SU(2)$ and $G = SO(3)$. When $M/G = S^3$, we show that there are at most 3 singular orbits. In all three cases, we describe the structure of singular orbit strata, which allows us to glue different pieces of singular orbits to recover the topology of M if exceptional orbits do not occur. If exceptional orbits occur, we show that the stratification of M/G must be very special.

We note that Fuquan Fang also studied the classification of positively curved 6-manifolds with symmetry group containing $SU(2)$ or $SO(3)$. In addition, we discuss the case of finite non-trivial isotropy groups and the case when the orbit space M/G is homeomorphic to B^3 or B^4. We also point out that there are in fact examples for which $G = SO(3)$, $M^G \neq \emptyset$ but M is not diffeomorphic to S^6. See Section 5, Example 2(c).

This paper is organized as follows: Section 2 is devoted to preliminary knowledge on group actions and manifolds with positive sectional curvature. In Section 3 we study different orbit types for all possible actions considered in this paper. In Section 4 we prove Theorem 1.1, Theorem 1.2 and Theorem 1.3, and study actions with more complicated orbit stratification. In Section 5 we describe all known examples of isometric $SU(2)$- and $SO(3)$-actions on positively curved 6-manifolds.

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2. Preliminaries

We start by recalling some basic definitions for group actions, see e.g. [Bre72] and [AB15] for a reference. Let G be a compact Lie group and M be a compact smooth manifold. For a smooth action $\pi : G \times M \to M$, the *G-orbit* $G \cdot p$ through a point $p \in M$ is the submanifold $G \cdot p = \{gp \in M | g \in G\}$, the *isotropy group* or the stabilizer at $p \in M$ is defined as $G_p = \{g \in G | gp = p\}$, and we have $G \cdot p = G/G_p$. Furthermore, we denote the *G-fixed point set* by $M^G = \{p \in M | G \cdot p = p\}$. Note also that the fixed point set in an orbit has the form $(G/K)^H = \{g \in G | g^{-1}Hg \subset K\}/K$, where $H \subset K \subset G$. In particular, $(G/H)^H = N(H)/H$.

Points in the same G-orbits have conjugate isotropy groups. The *isotropy type* of a G-orbit G/H is the conjugacy class of isotropy groups at points in G/H and denote it by (H). We define $M_{(K)}$ to be the union of orbits with the same isotropy type (K). For compact group actions on compact manifolds, there are only finitely many orbit types.

Among all orbit types of a given action, there exist maximal orbits G/H with respect to inclusion of isotropy groups called the *principal orbits*. Non-principal orbits which have the same dimension as the principal orbit are called *exceptional orbits*, and orbits having lower dimension than principal ones are called *singular orbits*.

The *orbit space* $M^* = M/G$ is the union of its *orbit strata* $M^*_{(K)} = M_{(K)}/G$ which themselves are manifolds. The principal orbit stratum $M^*_{(H)}$ is an open, dense and connected subset of M/G. In particular the dimension of $M^*_{(H)}$ is called the *cohomogeneity* of the action. Codimension one strata in M^* are called *faces*, which are part of ∂M^*. We will also use the fact that $M^K_{(K)} \to M^*_{(K)}$ is an $N(K)/K$-principal bundle and the structure group of $M_{(K)} \to M^*_{(K)}$ is $N(K)/K$.

The following theorem gives constraints on the exceptional orbits on simply-connected manifolds:

Theorem 2.1. *(*[Bre72]*, Theorem IV.3.12) Let M be a simply-connected manifold and G a compact group acting on M. Then M^* is also simply connected and there are no exceptional orbits G/K whose stratum $M^*_{(K)}$ has codimension* 1 *in M^* (so called special exceptional orbits).*

The following theorem describes the manifold structure on M^* for certain actions of cohomogeneity 3.

Theorem 2.2. *(*[Bre72]*, Corollary IV.4.7) Let M be a compact and simply-connected manifold and G a compact group acting on N. Suppose that all orbits are connected. Then M^* is a simply connected topological* 3*-manifold with or without boundary.*

For each orbit $G \cdot p$, let $T_p^\perp$ denote the normal space at p to the orbit and ν_p the unit sphere in the normal space. $T_p^\perp$ admits a natural linear action by the isotropy group G_p, called the *slice representation*. The quotient $T_p^\perp/G_p$ is called the *tangent cone* of the orbit in the orbit space, and ν_p/G_p is the *space of directions* at p and is denoted as $\Sigma_{[p]}$. If G acts on M by isometries, the orbit space, tangent cones and spaces of directions all inherit a metric from M. In particular, if we impose the positive curvature assumption on M, M/G becomes an Alexandrov space with positive curvature.

We also note that $M_{(K)} \cap T_p^\perp = (T_p^\perp)^K$, and the slice theorem states that an equivariant neighborhood of G/G_p has the form $G \times_{G_p} D(T_p^\perp) = (G \times D(T_p^\perp))/G_p$, where $D(T_p^\perp)$ is a disk in $T_p^\perp$ (also called the *slice* at p) and G_p acts on G via right multiplication and on $D(T_p^\perp)$ via the slice representation.

We now state the Extent Lemma. For any metric space (X, d) and positive integer $q \geq 2$, we define the *q-extent* of X as

$$xt_q(X) = \frac{1}{\binom{q}{2}} \sup_{x_1,\ldots,x_q \in X} \sum_{1 \leq i < j \leq q} d(x_i, x_j). \tag{2.1}$$

In other words, $xt_q(X)$ is the maximal average distance between points in q-touples in X. When $q = 2$, $xt_2(X)$ is the diameter of X. The Extent Lemma from [GS97] states that:

Lemma 2.3. *If M/G is an Alexandrov space with positive curvature, then for all (q+1)-touples $([x_0], ..., [x_q])$ in M/G, we have:*

$$\frac{1}{q+1}\sum_{i=0}^{q} xt_q(\Sigma_{[x_i]}) > \frac{\pi}{3}.$$

For the Euler Characteristic we have

Theorem 2.4. *(a)* [Kob58] *If a torus T acts smoothly on a closed smooth manifold M, then the Euler characteristic of M equals that of M^T, that is, $\chi(M) = \chi(M^T)$;*

(b) [PS02] *If M is a 6-dimensional simply connected Riemannian manifold with positive sectional curvature and S^1-symmetry, then $\chi(M)$ is positive and even.*

For totally geodesic submanifolds of positively curved manifolds, we have the so called Connectedness Lemma due to Burkhard Wilking:

Theorem 2.5. *(Connectedness Lemma,* [Wil03]*) Let M^n be a compact n-dimensional Riemannian manifold with positive sectional curvature. Suppose that $N^{n-k} \subset M^n$ is a compact totally geodesic embedded submanifold of codimension k. Then the inclusion map $N^{n-k} \hookrightarrow M^n$ is (n-2k+1)-connected.*

Recall that if we have a continuous map $f : X \to Y$ between two connected topological spaces X and Y, and a positive integer k, then we say that f is k-connected if f_* induces isomorphisms on homotopy groups $\pi_i,\ 1 \le i \le k-1$ and surjection on π_k.

For smooth actions on positively curved mainifolds with nontrivial principal isotropy group, we have:

Theorem 2.6. *(Isotropy Lemma,* [Wil06]*) Let G be a compact Lie group acting isometrically and not transitively on a positively curved manifold (M, g) with nontrivial principal isotropy group H. Then any nontrivial irreducible subrepresentation of the isotropy representation of G/H is equivalent to a subrepresentation of the isotropy representation of K/H, where K is an isotropy group such that the orbit stratum of K is a boundary face in M/G and K/H is a sphere.*

Let G be a compact Lie group and M be a closed smooth manifold. For a smooth G-action on M with non-empty fixed point set M^G, we define the *fixed point cohomogeneity* of this action as

$$\text{cohomfix}(M, G) = \dim(M/G) - \dim(M^G) - 1, \tag{2.2}$$

where $\dim(M^G)$ is the dimension of the fixed point component of largest dimension.

For Riemannian manifolds with positive sectional curvature and low fixed point cohomogeneity, we have the following classification.

Theorem 2.7. [GS97][GK04] *If M is a positively curved simply connected closed manifold which admits an isometric action by a compact group G such that the fixed point cohomogeneity is less than or equal to 1, then M is equivariantly diffeomorphic to a compact rank one symmetric space with an isometric G-action.*

By examining the actions on rank one symmetric spaces in Section 5, it follows that

Corollary 2.8. *Let M be a 6-dimensional positively curved simply connected close manifold which admits an isometric action by $G = SU(2)$ or $SO(3)$. If the action is fixed point homogeneous, then it is given by Examples 1(a), 1(b), or 2(a). If the action has fixed point cohomogeneity one, then it is given by Example 1(c).*

We now state a version of the soul theorem in the setting of orbit spaces.

Theorem 2.9. *(Theorem 1.2* [GK04]*, boundary soul lemma) Let M be a closed Riemannian manifold with positive sectional curvature and G a compact Lie group acting isometrically on M. Suppose $M^* = M/G$ has nonempty boundary ∂M^*. Then we have*

(1) There exists a unique point $s_o \in M^*$, the soul of M^*, at maximal distance to ∂M^*;
(2) The space of directions $S_{[s_o]}$ at s_o is homeomorphic to ∂M^*;
(3) The strata in int$(M^*) = M^* - \partial M^*$ belong to one of the following:
  (a) all of int(M^*);
  (b) the soul point s_o;
  (c) a cone over strata in ∂M^* with its cone point s_o removed;
  (d) a stratum containing s_o in its interior and whose boundary consists of strata in ∂M^*.

Remark 2.10. We point out that the regular points in M^* belong to either (a) or (c) in part (3). We also note that in [GK04] it was claimed that the strata in part (d) is one-dimensional, but one easily gives examples where its dimension is higher. We give an example here. Consider the Hopf action by S^1 on S^3, and the $SO(3)$ action on S^5 given by the restriction of the diagonal action of $SO(3)$ on $S^5 \subset \mathbb{R}^3 \oplus \mathbb{R}^3$. Note that $S^5/SO(3)$ is a 2-dim hemisphere of radius $\frac{1}{2}$, which we denote by H. Now consider the product sum action by $S^1 \times SO(3)$ on $S^3 * S^n$, where $*$ is the spherical join. The orbit space is then $X = S^2(\frac{1}{2}) * H$, where $S^2(\frac{1}{2})$ is the 2-dimensional sphere of radius $\frac{1}{2}$. The boundary is $\partial X = S^2 * \partial H = S^2(\frac{1}{2}) * S^1(\frac{1}{2})$. The soul point of X is the soul point of H. The strata of X are: $S^2(\frac{1}{2})$, $\text{int}(H)$, ∂H, $\partial H * S^2(\frac{1}{2}) - (\partial H \cup S^2(\frac{1}{2}))$, and $\text{int}(H) * S^2(\frac{1}{2}) - (\text{int}(H) \cup S^2(\frac{1}{2}))$. The rest are the regular points. In particular, the soul lies in a 2-dim strata.

We frequently use the knowledge of the subgroups of $SO(3)$ and $SU(2)$. For $SO(3)$ they are given by

- 0-dimensional subgroups: $\mathbb{Z}/k\mathbb{Z}$, D_k (dihedral groups acting on k vertices), A_4, S_4, A_5;
- 1-dimensional subgroups: $SO(2)$, $O(2)$;

and for $SU(2)$ by

- 0-dimensional subgroups: $\mathbb{Z}/k\mathbb{Z}$, binary dihedral groups, inverse images of A_4, S_4, A_5 in $SU(2)$;
- 1-dimensional subgroups: $U(1)$, $Pin(2) = N(U(1))$.

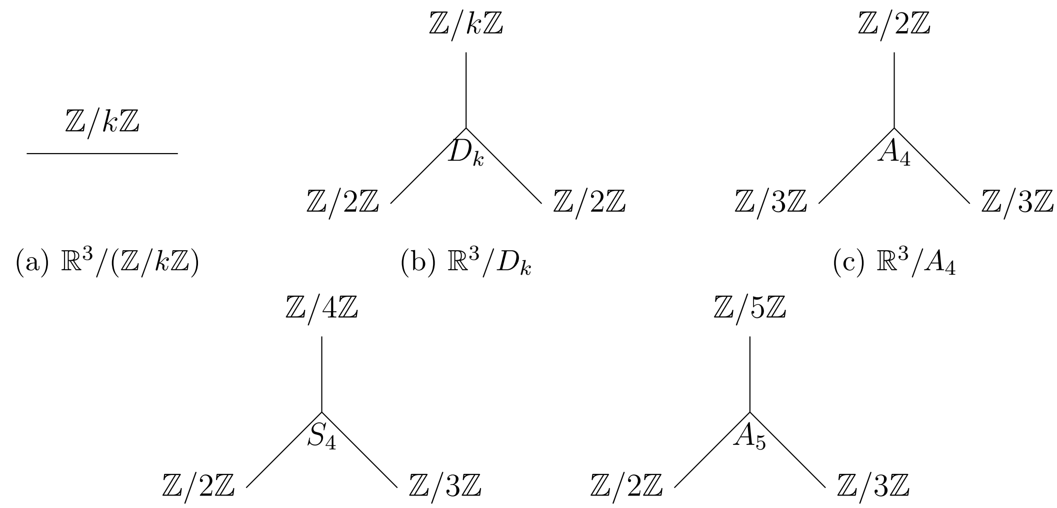
Note that the only subgroups of $SU(2)$ which do not contain the center $\mathbb{Z}/2\mathbb{Z}$ are cyclic groups of odd order.

It will also be useful for us to describe the quotient of $\mathbb{R}^3$ under a finite subgroup Γ of $SO(3)$. In Figure 1, a line segment represents a stratum of $\mathbb{R}^3/\Gamma$ with indicated cyclic isotropy, the origin has isotropy Γ and the complement has trivial isotropy.

We also describe effective $O(2)$-representations on $\mathbb{R}^4$.

Theorem 2.11. *Let $\rho : O(2) \to O(4)$ be an effective representation of $O(2)$ and $SO(2) \subset O(2)$ act as $\rho(R(\theta)) = \begin{bmatrix} R(p\theta) & 0 \\ 0 & R(q\theta) \end{bmatrix}$, where $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is the rotation matrix, and p, q are coprime integers. Take $\tau \in O(2) \setminus SO(2)$ such that $\tau^2 = id$. Then up to conjugation and permutation of p and q, ρ lies in one of the following categories:*

(a) $p, q \neq 0$, and $\rho(\tau) = diag(1, -1, 1, -1)$;
(b) $p = 0, q = 1$, and $\rho(\tau) = diag(1, 1, 1, -1)$;
(c) $p = 0, q = 1$, and $\rho(\tau) = diag(-1, -1, 1, -1)$.



(a) $\mathbb{R}^3/(\mathbb{Z}/k\mathbb{Z})$ (b) $\mathbb{R}^3/D_k$ (c) $\mathbb{R}^3/A_4$ (d) $\mathbb{R}^3/S_4$ (e) $\mathbb{R}^3/A_5$

Figure 1. Finite quotients of $\mathbb{R}^3$

3. The Structure of Orbit Spaces and Orbit Types

Throughout the remainder of the paper, G will always be the Lie group $SU(2)$ or $SO(3)$, and M is a simply connected closed 6-dimensional Riemannian manifold with positive sectional curvature which admits an effective isometric G-action.

We start by observing the following dichotomy for the topology of the orbit space M/G:

Theorem 3.1. *The orbit space M^* is homeomorphic to either S^3, or a 3-ball B^3, or B^4. When $M^* = B^4$, M is equivariantly diffeomorphic to S^6 with a fixed point homogeneous linear $SO(3)$-action.*

Proof. The cohomogeneity is calculated via $\dim(M^*) = \dim(M) - \dim(G) + \dim(H) = 3 + \dim(H)$, where H is the principal isotropy group. H is either 0 or 1-dimensional, since closed subgroups of $G = SU(2)$, $SO(3)$ have dimensions 0,1,3, and H cannot be 3-dimensional since otherwise the G-action would be trivial. Thus the cohomogeneity is either 3 or 4.

Suppose that the cohomogeneity is 4. Then the principal isotropy group H is 1-dimensional, thus one of S^1, $O(2)$ or $Pin(2)$. Since the isotropy representation of G/H is irreducible, Theorem 2.6 implies that the isotropy representation of G/H is equivalent to a subrepresentation of the isotropy representation of K/H where $K \subset G$ is an isotropy group of a boundary face. The only possibility is $K = G$ and thus one boundary face has isotropy $K = G$, which means that the G-action is fixed point homogeneous. From Corollary 2.8, it follows that the only fixed point homogeneous action of G on M with cohomogeneity 4 is the linear $SO(3)$-action on S^6 described in Example 1(b) in Section 5.

When the cohomogeneity is 3, Theorem 2.2 implies that the orbit space M^* is a simply connected 3-dimensional topological manifold possibly with boundary. If ∂M^* is non-empty, Theorem 2.9(1) implies that M^* is a 3-ball since the soul is a point. If ∂M^* is empty, then M^* is a simply connected 3-manifold without boundary, thus a 3-sphere by Perelman's solution to the Poincare conjecture [Per03]. $\square$

We note that we have 4 kinds of orbits, corresponding to the 0-,1-, or 3-dimensional closed subgroups of G:

(a) Principal orbits G/H, with principal isotropy group H, which will be shown to be trivial, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, or $SO(2)$;
(b) Exceptional orbits G/Γ, with isotropy groups Γ, which are finite extensions of H; we will show Γ is cyclic or dihedral when H is trivial;
(c) Singular orbits G/K, with 1-dimensional isotropy groups K, and hence $K = SO(2),\ O(2)$ when $G = SO(3)$, and $K = U(1),\ Pin(2)$ when $G = SU(2)$;
(d) Fixed points, i.e. $G_p = G$.

We start by showing that we can assume the principal isotropy is trivial.

Theorem 3.2. *If the principal isotropy subgroup H is non-trivial, then M is equivariantly diffeomorphic to S^6 with a linear $SO(3)$-action with fixed point cohomogeneity at most 1.*

Proof. We break up the proof into several lemmas.

Lemma 3.3. *If $M^G \neq \emptyset$ and the principal isotropy H is non-trivial, then the G-action on M has fixed point cohomogeneity at most 1.*

Proof. We separate the cases of $SU(2)$ and $SO(3)$ actions.

- Case 1: $G = SU(2)$.
  $SU(2)$ acts on the normal space to M^G effectively without fixed points. By considering the faithful real representations without trivial summands of $SU(2)$ in dimensions less than 6, we see that only the 4-dimensional irreducible representation of $SU(2)$ satisfies the requirements. This representation is the realification of the standard $SU(2)$-action on $\mathbb{C}^2$, and hence M is fixed point homogeneous. But then H is trivial, contradicting our assumption.
- Case 2: $G = SO(3)$.
  For the action of $SO(3)$ on the normal space to M^G we have the following possibilities:
  (a) $\mathbb{R}^3$ with the standard $SO(3)$-action. In this case the action is fixed point homogeneous;
  (b) $\mathbb{R}^5$ with the unique 5 dim irreducible representation of $SO(3)$. The $SO(3)$-action on the unit normal sphere S^4 has cohomogeneity one, which by definition implies the G-action on M has fixed point cohomogeneity one;
  (c) $\mathbb{R}^3 \oplus \mathbb{R}^3$ with diagonal action of $SO(3)$. But in this case the principal isotropy is trivial, since $(\vec{x}, \vec{y}) \in \mathbb{R}^3 \oplus \mathbb{R}^3$ has trivial isotropy when $\vec{x}$ and $\vec{y}$ are linearly independent.

□

From Corollary 2.8 the only actions with fixed point cohomogeneity at most one and with non-trivial principal isotropy are the $SO(3)$-actions on S^6 described in Examples 1(b) and 1(c) in Section 5.

In Lemma 3.4 and Lemma 3.5 we will show that the case of $M^G = \emptyset$ does not occur when the principal isotropy H is non-trivial.

Lemma 3.4. *If $M^G = \emptyset$ and H is non-trivial, then $G = SO(3)$, $H = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and ∂M^* consists of one face with isotropy $O(2)$. Furthermore, the interior has at most one singular orbit.*

Proof. By Theorem 3.1, we only need to consider the case $\dim(M^*) = 3$ and thus H is finite. First notice that the Isotropy Lemma implies that $\partial M^* \neq \emptyset$, and thus $M^* = B^3$. If the isotropy representation of H on the tangent space to G/H has an irreducible subrepresentation of dimension greater than 1, then by the Isotropy Lemma, the isotropy K of the boundary face has dimension at least 2. Hence $K = G$, i.e. the G-action is fixed point homogeneous, which contradicts our assumption of $M^G = \emptyset$.

In all other cases the irreducible components of the (3-dimensional) isotropy representation of G/H are 1-dimensional. Among the non-trivial subgroups of G, only $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ in $SO(3)$ has 3-dimensional representations of this type. Thus $H = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $G = SO(3)$. Since the isotropy representation of $SO(3)/H$ has a 1-dimensional subrepresentation on which H acts as -Id, the Isotropy Lemma implies that the boundary face has isotropy $K = O(2)$. The only strata with higher-dimensional isotropy groups on ∂M^* are fixed points, which cannot occur by our assumption.

So we obtain that the boundary of the orbit space has isotropy $O(2)$ and hence the interior regular part has isotropy $H = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. From Theorem 2.9(3), it follows that M^* has at most one interior singular orbit which would have to be the soul point. □

In the following lemma, we will rule out both possibilities on the number of interior singular orbits by calculating the fundamental group and cohomology groups of M.

Lemma 3.5. *If M^G is empty and there are no interior singular orbits, then M is not simply connected. Moreover, there are no actions with one interior singular orbit.*

Proof. Recall that according to Lemma 3.4, we have $G = SO(3)$. From now on, $\pi : M \to M^*$ is the natural projection, $U = \pi^{-1}(\mathrm{int}(M^*))$ and V is a tubular neighborhood of $\pi^{-1}(\partial M^*)$. $M = U \cup V$ is the desired decomposition.

(a) If M^* has no interior singular orbit, then U is an $SO(3)/H$-bundle over $\mathrm{int}(M^*) = D^3$ and thus $U = SO(3)/H \times D^3$ since the base D^3 is contractible. V deformation retracts onto an $SO(3)/O(2)$-bundle over $\partial M^* = S^2$, with structure group $N(O(2))/O(2) = id$. Thus V retracts onto a trivial $\mathbb{RP}^2$-bundle over S^2. $U \cap V$ deformation retracts onto $SO(3)/H \times S^2$. Let $i : U \cap V \to U$, $j : U \cap V \to V$ denote the respective inclusions. By van Kampen's theorem, $\pi_1(M) = \pi_1(U) * \pi_1(V)/ < i_*(a)j_*(a)^{-1} | a \in \pi_1(U \cap V) >$. It is non-trivial since i_* is an isomorphism and thus $\pi_1(V) \cong \mathbb{Z}/2\mathbb{Z}$ cannot be killed. This contradicts the assumption that M is simply connected.

(b) If M^* has an interior singular orbit, a priori the singular orbit could have isotropy $SO(2)$ or $O(2)$. Recall that for the Euler characteristic we have $\chi(U \cup V) = \chi(U) + \chi(V) - \chi(U \cap V)$ by [Bre72]. Take the $SO(2)$-fixed point set $M^{SO(2)} = V^{SO(2)} \cup U^{SO(2)}$. Since $SO(2)$ fixes one point in each orbit over ∂M^*, $V^{SO(2)}$ is homeomorphic to $\partial M^* = S^2$. If the interior singular orbit has isotropy $O(2)$, then $U^{SO(2)} = (SO(3)/O(2))^{SO(2)} = \mathrm{pt}$. So Theorem 2.4 implies that $\chi(M) = \chi(M^{SO(2)}) = \chi(S^2 \cup \mathrm{pt}) = 3$, contradicting the fact that $\chi(M)$ is even by Theorem 2.4. Thus the interior singular orbit has isotropy $SO(2)$. Suppose that the slice representation of $SO(2)$ has slope (p, q), i.e. it acts on $\mathbb{C} \oplus \mathbb{C}$ as $(z_1, z_2) \mapsto (\xi^p z_1, \xi^q z_2)$ for $|\xi| = 1$. Then $H = \mathbb{Z}/\gcd(p, q)\mathbb{Z}$, and in particular H is cyclic. Thus $H = \mathbb{Z}/2\mathbb{Z}$ and $U^{SO(2)} = (SO(3)/SO(2))^{SO(2)} = 2$ points. Therefore $\chi(M) = \chi(M^{SO(2)}) = \chi(S^2 \cup \{2 \text{ points}\}) = 4$. We first show that M has the cohomology groups of $\mathbb{CP}^3$, and then use the Mayer-Vietoris sequence to obtain a contradiction.

M^H is a totally geodesic submanifold with even codimension since the H-action on M is orientation-preserving, and it has dimension at least 3 since H fixes at least one point in each principal orbit. Thus M^H is a 4-dimensional submanifold with positive sectional curvature which is connected by Frankel's theorem, see [Fra61]. Wilking's connectedness lemma implies that the inclusion of $i : M^H \hookrightarrow M$ is 3-connected. In particular, $\pi_1(M^H) = \pi_1(M) = 0$. Moreover, $N(H)/H = O(2)$ acts effectively on M^H. M^H is homeomorphic to S^4 or $\mathbb{CP}^2$ by [HK89]. If $M^H = S^4$, then Wilking's connectedness lemma implies M is a homology 6-sphere, violating $\chi(M) = 4$. Thus $M^H = \mathbb{CP}^2$. We then have $\pi_2(M) = \pi_2(\mathbb{CP}^2) = \mathbb{Z}$, $\pi_3(M) = i_*(\pi_3(\mathbb{CP}^2)) = 0$.

Hence the Hurewicz theorem and Poincare duality imply that M has the cohomology groups of $\mathbb{CP}^3$.

Now we apply the Mayer-Vietoris sequence to compute $H^*(M)$ again. From the slice theorem, $U = SO(3) \times_{SO(2)} D^4$. U deformation retracts onto S^2. V is an $SO(3)/O(2)$-bundle over S^2 with structure group $N(O(2))/O(2) = \mathrm{id}$ and thus V is homotopy equivalent to $S^2 \times \mathbb{RP}^2$. $U \cap V$ is homotopy equivalent to $\partial U = SO(3) \times_{SO(2)} S^3$. By lifting the action of $SO(2)$ on $SO(3) \times S^3 = \mathbb{RP}^3 \times S^3$ to an action on $S^3 \times S^3$, we obtain $\partial U = (S^3 \times S^3)/S^1$, where S^1 acts on $S^3 \times S^3$ as $x \cdot (p, q) = (x^k p, x^l q),\ x \in S^1,\ (p, q) \in S^3 \times S^3,\ k, l \in \mathbb{Z}$. Here we view x as a unit complex number and $p,\ q$ as unit quaternions, and the multiplication is quaternionic multiplication. By Proposition 2.3(a) in [WZ90], it follows that ∂U is diffeomorphic to $S^3 \times S^2$. We then have the following short exact sequence:

$$0 \to H^2(M) \cong \mathbb{Z} \to H^2(U) \oplus H^2(V) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to H^2(U \cap V) \cong \mathbb{Z} \to H^3(M) = 0, \tag{3.1}$$

which leads to a contradiction.

□

Combining Lemmas 3.3, 3.4 and 3.5 finishes the proof of Theorem 3.2. □

In our setting, exceptional orbits could have rich and complicated structure, as we will see in the next sections, making it difficult to recover the original manifold from the orbit space. We state and prove some results on the structure of exceptional orbits, including Theorem 1.1(c).

Proposition 3.6. *For exceptional orbits G/Γ, the following holds:*

(a) The exceptional isotropy groups Γ are cyclic of odd order if $G = SU(2)$, and cyclic or dihedral if $G = SO(3)$;

(b) In the orbit space M/G, there are no exceptional strata whose closure does not contain singular strata.

Proof. We prove part (a) via case-by-case analysis. The finite subgroups of $SO(3)$ are $\mathbb{Z}/m\mathbb{Z},\ D_n,\ A_4,\ S_4,\ A_5$.

If $\Gamma = A_4$, then the local picture of the exceptional strata is Figure 1c. We take the $\mathbb{Z}/2\mathbb{Z}$-fixed point set, and note that

$$\begin{aligned}(SO(3)/A_4)^{\mathbb{Z}/2\mathbb{Z}} &= \{g \in SO(3) | g^{-1}(\mathbb{Z}/2\mathbb{Z})g \subset A_4\}/A_4 = N(\mathbb{Z}/2\mathbb{Z})A_4/A_4 \\ &= N(\mathbb{Z}/2\mathbb{Z})/N(\mathbb{Z}/2\mathbb{Z}) \cap A_4 = N(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}) = S^1 \coprod S^1, \\ (SO(3)/(\mathbb{Z}/2\mathbb{Z}))^{\mathbb{Z}/2\mathbb{Z}} &= N(\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}) = S^1 \coprod S^1,\ (SO(3)/(\mathbb{Z}/3\mathbb{Z}))^{\mathbb{Z}/2\mathbb{Z}} = \emptyset.\end{aligned}$$

We conclude that $M^{\mathbb{Z}/2\mathbb{Z}}$ has two circle boundaries at $SO(3)/A_4$. On the other hand, each component of $M^{\mathbb{Z}/2\mathbb{Z}}$ is a 2-sphere, which is a contradiction.

If $\Gamma = S_4$, then the local picture of the exceptional strata is Figure 1d. Considering $M^{\mathbb{Z}/3\mathbb{Z}}$ we have

$$(SO(3)/S_4)^{\mathbb{Z}/3\mathbb{Z}} = N(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}) = S^1 \coprod S^1,\ (SO(3)/(\mathbb{Z}_3\mathbb{Z}))^{\mathbb{Z}/3\mathbb{Z}} = N(\mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/3\mathbb{Z}) = S^1 \coprod S^1,$$
$$(SO(3)/(\mathbb{Z}/2\mathbb{Z}))^{\mathbb{Z}/3\mathbb{Z}} = \emptyset,\ (SO(3)/(\mathbb{Z}/4\mathbb{Z}))^{\mathbb{Z}/3\mathbb{Z}} = \emptyset,$$

which again leads to a contradiction since $M^{\mathbb{Z}/3\mathbb{Z}} = S^2$.

If $\Gamma = A_5$, then the local picture of the exceptional strata is Figure 1e. Considering $M^{\mathbb{Z}/5\mathbb{Z}}$ we have

$$(SO(3)/A_5)^{\mathbb{Z}/5\mathbb{Z}} = N(\mathbb{Z}/5\mathbb{Z})/(\mathbb{Z}/5\mathbb{Z}) = S^1 \coprod S^1,\ (SO(3)/(\mathbb{Z}/5\mathbb{Z}))^{\mathbb{Z}/5\mathbb{Z}} = N(\mathbb{Z}/5\mathbb{Z})/(\mathbb{Z}/5\mathbb{Z}) = S^1 \coprod S^1,$$

$(SO(3)/(\mathbb{Z}/2\mathbb{Z}))^{\mathbb{Z}/5\mathbb{Z}} = \emptyset,\ (SO(3)/(\mathbb{Z}/3\mathbb{Z}))^{\mathbb{Z}/5\mathbb{Z}} = \emptyset,$

again a contradiction.

In conclusion, $\Gamma \neq A_4,\ S_4,\ A_5$, and hence the exceptional isotropy groups are cyclic or dihedral.

To prove part (b), we first observe that exceptional orbit strata cannot be 2-dimensional, as there are no special exceptional orbits. Thus they are isolated points or 1-dimensional curves. We want to show that there are no connected components of exceptional strata whose closure does not contain singular orbits, in particular, exceptional points can not be isolated.

Suppose there is a component of exceptional strata which is closed. Then it is a connected graph, which is a union of circles and intervals. Take some exceptional isotropy group Γ of the strata and a non-trivial cyclic subgroup C of Γ, and consider the C-fixed point component in this exceptional stratum.

In each exceptional orbit G/Γ, $(G/\Gamma)^C$ is the union of several circles. Since the exceptional strata are 1-dim, the fixed point component is 2-dim, and hence is a 2-sphere, as it is orientable, totally geodesic and hence has positive curvature. This induces a foliation of S^2 by circles, which is impossible since the tangent bundle of S^2 does not contain any sub line bundle. □

4. Actions, Orbit Spaces and the Topology of G-manifolds

In this section we study different types of G-actions on positively curved 6-manifolds. We start with the case of non-empty fixed point set.

Proposition 4.1. *If $M^G \neq \emptyset$, then one of the following holds:*

(a) M is equivariantly diffeomorphic to S^6 or $\mathbb{CP}^3$ with a linear action;

(b) $G = SO(3)$ and M^G is finite. In this case $M^* = B^3$ and M^G lies on ∂M^*.

Proof. We first note that a substantial part of the proof has already appeared in the proof of Lemma 3.3. In fact, we proved that if $G = SU(2)$, then M is fixed point homogeneous. If $G = SO(3)$, either M has fixed point cohomogeneity at most one, which are classified by Theorem 2.7, or the action on the tangent space at fixed points is given by $A \cdot (\vec{x}, \vec{y}) = (A\vec{x}, A\vec{y}),\ A \in SO(3),\ (\vec{x}, \vec{y}) \in \mathbb{R}^3 \oplus \mathbb{R}^3$. Thus the fixed points are isolated and the orbit types of this action are:

- principal orbits with trivial isotropy, represented by two linearly independent vectors in $\mathbb{R}^3$;
- singular orbits with $SO(2)$-isotropy, represented by two linearly dependent vectors in $\mathbb{R}^3$ which are not both zero;
- the fixed point $(0, 0)$.

The union of singular orbits near a fixed point has dimension 4 in M, which descends to 2-dimensional strata of M^*. Since $\dim(M^*) = 3$, this strata is a boundary face in ∂M^* and hence $M^* = B^3$ by Theorem 3.1. Moreover, from the above discussion, all fixed points are contained in the boundary strata.

□

Next, we study the case where M^G is finite or empty.

4.1. $SO(3)$ **actions with** M^G **finite or empty.** By Theorem 3.1, we divide this section into two parts, corresponding to $M^* = B^3$ and $M^* = S^3$, and we start with the case where $M^* = B^3$.

Theorem 4.2. *Assume $G = SO(3)$, with $M^* = B^3$ and M^G finite. Then:*

(a) The principal isotropy is trivial and the boundary face of M^* consists of singular orbits with $SO(2)$-isotropy.
(b) If ∂M^* contains more than 1 orbit type, then ∂M^* contains exactly 2 singular points which are either two fixed points or one fixed point and one $O(2)$-orbit. Furthermore, these two singular orbits do not lie in a face.
(c) If there is an $O(2)$-orbit on ∂M^*, then there exist an interior singular orbit and a $\mathbb{Z}/2\mathbb{Z}$-exceptional stratum connecting the interior singular orbit and the $O(2)$-orbit.
(d) There is at most 1 interior singular orbit whose isotropy group has to be $SO(2)$.
(e) The Euler characteristic $\chi(M) \leq 6$. If ∂M^* contains more than 1 orbit type, then $\chi(M) \leq 4$.

Proof. Proof of part (a): If the principal isotropy is non-trivial, then by Theorem 3.2 and Corollary 2.8, the action on M is given by Example 1(c) in Section 5. But then M^G is a circle, contradicting our assumption that M^G is finite. By Theorem 2.1 the boundary does not contain any exceptional orbits. If the boundary face orbits have $O(2)$-isotropy, then the slice action of $O(2)$ on the 4-dimensional normal space is either ineffective or the one given in Theorem 2.11(b) since the $O(2)$-strata is two-dimensional. In both cases the principal isotropy group would be non-trivial (containing $\mathbb{Z}/2\mathbb{Z}$).

Proof of part (d): Theorem 2.9(3) implies that $\mathrm{int}(M^*)$ has at most one isolated stratum which is the soul point. By Proposition 3.6(b), the soul point cannot be an exceptional orbit. Thus it has to be a singular orbit G/K. A priori K could be $SO(2)$ or $O(2)$. Suppose $K = O(2)$. The slice representation of $O(2)$ is 4-dimensional and orientation-reversing, since the isotropy representation of $O(2)$ on $SO(3)/O(2)$ is orientation-reversing. As in Theorem 2.11, let p and q denote the slopes of the $SO(2)$-subaction and $\tau \in O(2) \setminus SO(2)$. We list all possible effective orientation-reversing $O(2)$-actions on $\mathbb{R}^4$:

(1) $p, q \neq 0$, and τ acts by $\mathrm{diag}(1, -1, 1, -1)$. In this case the action of τ is orientation preserving, which implies that the slice action of $O(2)$ is orientation preserving. But this is not allowed.
(2) $p = 0, q = 1$, and τ acts by $\mathrm{diag}(1, 1, 1, -1)$. In this case the strata $M^*_{(O(2))}$ is 2-dimensional and thus $M^*_{(O(2))} \subset \partial M^*$, contradicting the assumption $G/K \in \mathrm{int}(M^*)$.
(3) $p = 0, q = 1$, and τ acts by $\mathrm{diag}(-1, -1, 1, -1)$. In this case $M^*_{(SO(2))}$ is 2-dimensional and $M^*_{(SO(2))} \subset \partial M^*$. G/K lies in the closure of $M^*_{(SO(2))}$, and thus $G/K \in \partial M^*$. Thus this also cannot occur.

In conclusion $K = SO(2)$.

Proof of part (c): If there is an $O(2)$-orbit on ∂M^*, then the slice action is effective since otherwise the principal isotropy is non-trivial. Note that the boundary face has isotropy $SO(2)$ and thus the slice action must be the $O(2)$-action given by Theorem 2.11(c), since it is the only case in which the $O(2)$-orbit has nearby $SO(2)$-strata. As a consequence the $O(2)$-orbits are isolated. Moreover by analyzing the action in Theorem 2.11(c), there exists a $\mathbb{Z}/2\mathbb{Z}$-stratum emanating from the $O(2)$-orbit. By Theorem 2.9(3) either this stratum ends at the soul point, or it ends at another boundary stratum point, which a priori could be an $O(2)$-orbit or a fixed point. However it cannot end at a fixed point, since by analyzing the orbit types near an isolated fixed point there are no $\mathbb{Z}/2\mathbb{Z}$-strata.

We claim that there cannot be two $O(2)$-orbits on ∂M^* and prove it by contradiction. Assume otherwise. Then we have the following possible cases:

(1) There exists a $\mathbb{Z}/2\mathbb{Z}$-stratum connecting the two $O(2)$-orbits. In this case, we take the fixed point set $M^{\mathbb{Z}/2\mathbb{Z}}$. First observe that for the boundary face orbits, $(SO(3)/SO(2))^{\mathbb{Z}/2\mathbb{Z}}$ =2 pt. For the $O(2)$-orbit, $(SO(3)/O(2))^{\mathbb{Z}/2\mathbb{Z}} = S^1\cup$ pt. For the $\mathbb{Z}/2\mathbb{Z}$-orbits, $(SO(3)/(\mathbb{Z}/2\mathbb{Z}))^{\mathbb{Z}/2\mathbb{Z}} = S^1 \cup S^1$. These are all the strata with non-empty $\mathbb{Z}/2\mathbb{Z}$-fixed point sets. On the other hand, $M^{\mathbb{Z}/2\mathbb{Z}}$ is an orientable 2-dim totally geodesic submanifolds with positive curvature, and thus $M^{\mathbb{Z}/2\mathbb{Z}}$ is the union of 2-spheres. The component of $M^{\mathbb{Z}/2\mathbb{Z}}$ over ∂M^* is a branched double cover of ∂M^*, and the other components are foliated by circles in $(SO(3)/(\mathbb{Z}/2\mathbb{Z}))^{\mathbb{Z}/2\mathbb{Z}}$ and $(SO(3)/O(2))^{\mathbb{Z}/2\mathbb{Z}}$. But 2-spheres cannot be foliated by circles. So this is not possible.

(2) There exist two $\mathbb{Z}/2\mathbb{Z}$-strata connecting the two $O(2)$-orbits to the soul point. Part (d) implies that the soul point has isotropy $SO(2)$. The 4-dim slice representation of $SO(2)$ is of the form $R(\theta) \mapsto \begin{bmatrix} R(2\theta) & 0 \\ 0 & R(\pm 2\theta) \end{bmatrix}$ since the nearby exceptional strata has $\mathbb{Z}/2\mathbb{Z}$ isotropy. However that forces the principal isotropy group to be non-trivial. So this is not possible either.

Proof of part (b): In each boundary face orbit, $SO(2)$ fixes exactly 2 points, since a boundary face orbit has $SO(2)$-isotropy; in $O(2)$-orbits on $\partial(M/G)$ or G-fixed points, $SO(2)$ fixes one point. Thus the $SO(2)$-fixed point component over $\partial(M/G)$ is a branched double cover of $\partial(M/G) = S^2$ with branch points corresponding to $O(2)$-orbits or G-fixed points. Moreover, the $SO(2)$-fixed point component is a 2-sphere itself, as it is orientable and has positive curvature. From the Riemann-Hurwicz formula, a branched double cover between two 2-spheres has exactly 2 branch points. We have shown in part (c) that there cannot be two $O(2)$-orbits. Thus the two branch points correspond to two fixed points or one fixed point and one $O(2)$-orbit.

Proof of part (e): If ∂M^* has more than 1 orbit type, then there are two singular points in ∂M^* by part (b) and at most one singular orbit in the interior which has $SO(2)$-isotropy. These include all singular orbits, and $M^{SO(2)}$ is either a 2-sphere or the union of a 2-sphere with 2 points. Thus $\chi(M) = \chi(M^{SO(2)}) \leq 4$. If ∂M^* has only 1 orbit type, then the subset of $M^{SO(2)}$ sitting over ∂M^* is the union of two 2-spheres since the whole boundary has $SO(2)$-isotropy and $SO(2)$ fixed two points in each boundary orbit. Moreover Theorem 2.9(3) implies that either $\mathrm{int}(M^*)$ is a stratum, or $\mathrm{int}(M^*) - s_0$ and the soul point are two strata. In other words, M^* has no exceptional orbits. Part (d) implies that the soul point has $SO(2)$-isotropy. Thus $M^{SO(2)} = S^2 \cup S^2$ or $S^2 \cup S^2 \cup \{2 \text{ points}\}$ and $\chi(M) = \chi(M^{SO(2)}) = 4,\ 6$.

□

Corollary 4.3. *If $G = SO(3)$ and $M^* = B^3$, then the structure of M^* is as in Figures 2a to 2f below. In the pictures, the groups represent the isotropy groups of the corresponding strata.*

Proof. If M^G is not finite, Proposition 4.1 implies that M^* is as in Figure 2a. So we may assume that M^G is finite. Then Theorem 4.2 implies that the boundary face has isotropy $SO(2)$ and that there is at most 1 interior singular orbit.

If ∂M^* has 1 orbit type and $\mathrm{int}(M^*)$ contains no singular orbit, M^* is as in Figure 2b. If $\mathrm{int}(M^*)$ contains 1 singular orbit, M^* is as in Figure 2c.

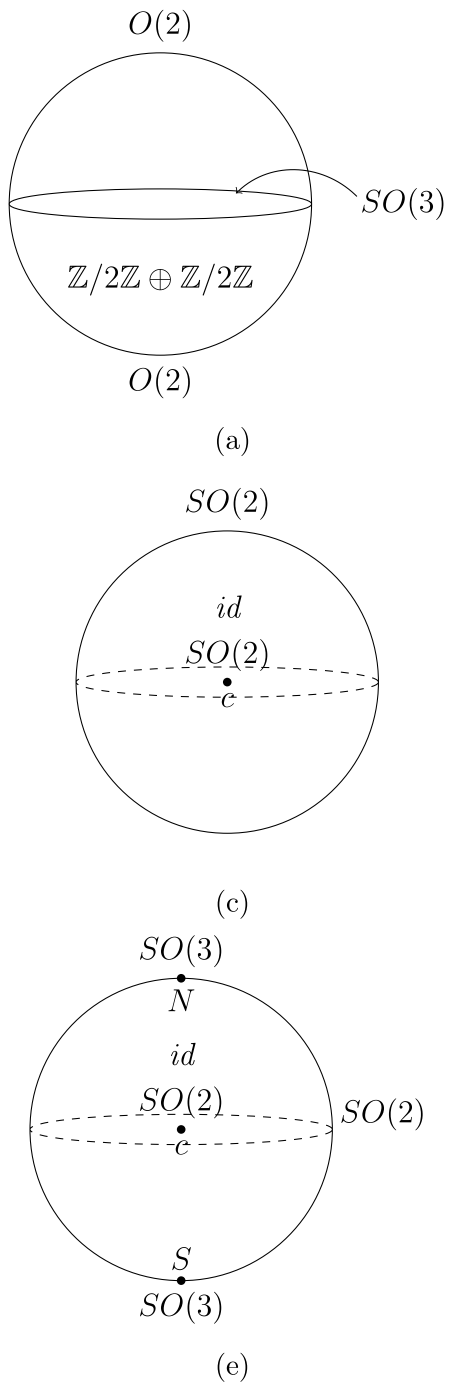


FIGURE 2. $G = SO(3),\ M^* = B^3$

If ∂M^* has multiple orbit types, Theorem 4.2 implies ∂M^* contains two fixed points, or one fixed point and one $O(2)$-orbit. If ∂M^* contains two fixed points, M^* is as in Figure 2d or Figure 2e. Otherwise, M^* is as in Figure 2f. $\square$

We point out that in Figures 2a and 2d, M is classified, see Proposition 4.1 and Theorem 4.4 respectively. We now prove Theorem 1.3.

Theorem 4.4. *Assume that $G = SO(3)$, $M^* = B^3$, ∂M^* contains more than 1 orbit type and that there are no exceptional orbits or interior singular orbits. Then M^6 is equivariantly homeomorphic to S^6 with a linear $SO(3)$-action as in Example 1(e).*

Proof. Theorem 4.2(b) implies that ∂M^* has 2 singular points. Since M has no exceptional orbits, these two orbits cannot be $O(2)$-orbits, otherwise there will be exceptional orbits near the $O(2)$-orbit with isotropy containing $\mathbb{Z}/2\mathbb{Z}$. Thus the two singular points are two G-fixed points. Now we see that the orbit types are: principal orbits with trivial isotropy in $\mathrm{int}(M^*)$, singular orbits on ∂M^* with $SO(2)$-isotropy and two G-fixed points on ∂M^*, and no interior singular orbits. We need to classify G-spaces with 3 orbit types $(H) = (id)$, $(K) = (SO(2))$, $(G) = (SO(3))$ such that the number of fixed points is 2.

We first recall Corollary V.6.2 in [Bre72]. For a smooth G-action on M, suppose the orbit space $X = M^*$ is a contractible manifold with boundary B, and that the action has only two orbit types, with principal orbits G/H corresponding to $X \setminus B$ and singular orbits G/K corresponding to B. Then the set of equivalence classes of such G-spaces is parametrized by the following set

$$[B,\ N(H)/(N(H) \cap N(K))]/\pi_0(\frac{N(H)}{H})$$

where $[X, Y]$ denotes the homotopy classes of continuous maps from X to Y.

For actions with 3 orbit types H, K and G, Proposition V.10.1 [Bre72] states that the set of equivariant homeomorphism classes of G-spaces with 3 orbit types is bijective to the set of equivariant homeomorphism classes of G-spaces with 2 orbit types (H) and (K) obtained by deleting the fixed points. The latter G-spaces are homotopy equivalent to G-spaces with orbit space a two-disk D^2 and 2 orbit types $H = id$, $K = SO(2)$ where the singular orbits G/K lie on the boundary of D^2. Those G-spaces are classified by

$$[\partial(D^2),\ N(H)/(N(H) \cap N(K))]/\pi_0(\frac{N(H)}{H}) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}.$$

But we can write down the two G-spaces. They are:

- the 5-sphere where the G-action comes from the restriction of the 6-dimensional real representation $\mathbb{R}^3 \oplus \mathbb{R}^3$ and $G = SO(3)$ acts diagonally;
- $S^2 \times S^3$ where $G = SO(3)$ acts diagonally on S^2-factor as the standard linear action and on S^3-factor as the linear suspension.

M is the suspension of the above 5-manifolds. But M is a manifold, so it can only be the suspension of the 5-sphere, which is a 6-sphere, and the action is the one described in Example 1(e).

□

Remark 4.5. The other cases of $G = SO(3)$ and $M^* = B^3$ listed in Corollary 4.3 are also interesting but not yet fully understood. For Figure 2b, ∂M^* has isotropy $SO(2)$ and $\mathrm{int}(M^*)$ has trivial isotropy. Corollary V.6.2 in [Bre72] implies that such G-spaces are parametrized by

$$[B,\ N(H)/(N(H) \cap N(K))]/\pi_0(\frac{N(H)}{H}) = \pi_2(\mathbb{RP}^2) = \mathbb{Z}.$$

Two examples of such G-spaces are $\mathbb{CP}^3$ and $S^2 \times S^4$. The $SO(3)$-action on $\mathbb{CP}^3$ is the one described in Example 2(b) in Section 5, while the action on $S^2 \times S^4$ is diagonal with standard SO(3)-action on S^2-factor and double suspension on S^4-factor. We suspect the G-spaces are oriented S^2-bundles over S^4, which are classified by the first Pontryagin class. We do not know whether $S^2 \times S^4$ with this $SO(3)$-action admits an invariant metric with positive sectional curvature. Of course, according to the Hopf conjecture, this should not have any metric with positive curvature at all.

For Figure 2f, Example 2(c) in Section 5 is an action with the corresponding orbit stratification. For Figures 2c and 2e, we do not have examples, and we expect that they actually do not exist.

We now turn to the case of $G = SO(3)$ and $M^* = S^3$.

Proposition 4.6. *If $G = SO(3)$ and $M^* = S^3$, then there are at most 3 singular orbits.*

Proof. There exist singular orbits since $\chi(M^{S^1}) = \chi(M) > 0$ and thus $M^{S^1} \neq \emptyset$. We apply the Extent Lemma to show that there are at most 3 singular orbits.

Suppose there were four singular orbits. Then in M^*, each singular orbit G/K_i has a space of direction $S^3(1)/K_i$, where $K_i = S^1$ or $O(2)$ acts linearly on the unit normal sphere $S^3(1)$ via the slice representation. Lemma 4 in [HK89] implies that $xt_3(S^3/K_i) \leq xt_3(S^2(\frac{1}{2})) = \frac{\pi}{3}$. Thus $\frac{1}{4}\sum_{i=0}^{3} xt_3(S^3(1)/K_i) \leq \frac{\pi}{3}$, and we get a contradiction to Lemma 2.3. $\square$

Remark 4.7. When $G = SO(3)$, $M^* = S^3$, the possible stratification of M^* is depicted in Figure 3. In

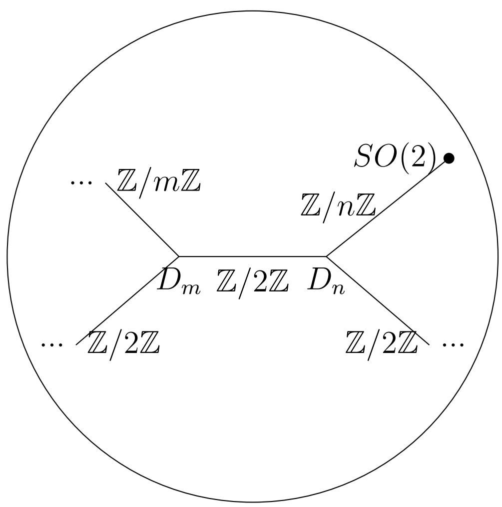


FIGURE 3. $G = SO(3)$, $M^* = S^3$

the picture, m, n are positive integers. Examples 2(d) and 3(b) in Section 5 are actions of this type.

4.2. SU(2) actions with $M^G = \emptyset$.

Proposition 4.8. *When $G = SU(2)$ and $M^G = \emptyset$, the orbit space M^* is a 3-sphere. Moreover, the fixed point set $M^{\mathbb{Z}/2\mathbb{Z}}$ of the center $\mathbb{Z}/2\mathbb{Z}$ is precisely the union of all singular orbits, which are all 2-spheres. Furthermore, there can be at most 3 singular orbits.*

Proof. We prove the second part first. Since the action of $\mathbb{Z}/2\mathbb{Z} \subset SU(2)$ preserves orientation, each component of $M^{\mathbb{Z}/2\mathbb{Z}}$ is a totally geodesic orientable submanifold of even codimension in M and $SO(3) =$

$SU(2)/(\mathbb{Z}/2\mathbb{Z})$ acts on it. A priori it could have dimension 4, 2, 0. But 0-dimensional components would be G-fixed points, violating our assumption. We now show that it cannot have dimension 4.

If a component of $M^{\mathbb{Z}/2\mathbb{Z}}$ has dim 4, then the induced metric has positive sectional curvature and is invariant under $SO(3)$. By Wilking's connectedness lemma, it is also simply connected, is diffeomorphic to either S^4 or $\mathbb{CP}^2$ by [HK89], and admits a cohomogeneity one action by $SO(3)$. From the classification of 4-dim cohomogeneity one manifolds (see for example [Par86]), such actions have at least one singular orbit with $O(2)$-isotropy, which lifts up to $Pin(2)$-isotropy for the corresponding $SU(2)$-action. The action of the $Pin(2)$-isotropy group on the normal space to $M^{\mathbb{Z}/2\mathbb{Z}}$ has to be effective, since otherwise the center $\mathbb{Z}/2\mathbb{Z}$ would lie in the ineffective kernel. But this is impossible since the normal space is 2-dimensional and $Pin(2)$ has no effective two-dim real representation. Thus every component of $M^{\mathbb{Z}/2\mathbb{Z}}$ is a 2-dimensional orientable positively curved manifold, which is a two-sphere. Those two-spheres are precisely the singular orbits, since every $U(1) \subset SU(2)$ contains $\mathbb{Z}/2\mathbb{Z}$ and hence every singular orbit is contained in $M^{\mathbb{Z}/2\mathbb{Z}}$.

Now we show $M^* = S^3$. Assume otherwise. Then by Theorem 3.1 $M^* = B^3$. But then ∂M^* consists of singular orbits, which means $M^{\mathbb{Z}/2\mathbb{Z}}$ is 4-dimensional since it contains all singular orbits, which is impossible.

If there are four singular orbits, then as in the proof of Proposition 4.6, we get a contradiction to Lemma 2.3. Thus there can be at most three. $\square$

Remark 4.9. The above proposition says more than the statement that the orbit space has no boundary. In fact, there are also no exceptional orbits whose isotropy groups contain the center $\mathbb{Z}/2\mathbb{Z}$, as a corollary. Hence the exceptional isotropy groups are all cyclic of odd order.

When $G = SU(2)$, and $M^* = S^3$, the possible stratifications are drawn in Figure 4. Here m, n, l

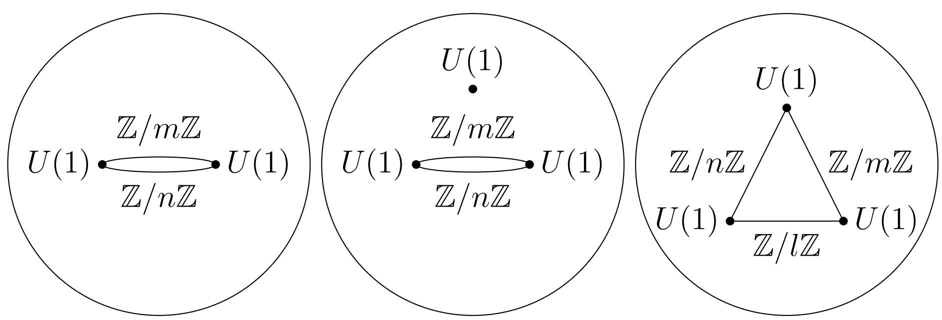


FIGURE 4. $G = SU(2),\ M^* = S^3$

are pairwise coprime odd integers. Example 4 in Section 5 is an action of this type.

We now prove Theorem 1.2.

Theorem 4.10. *$G = SU(2)$.*

(a) If the fixed point set M^G is non-empty, then M is equivariantly diffeomorphic to a linear action on S^6 or $\mathbb{CP}^3$.

(b) If M^G is empty and the action has no exceptional orbits, then M is diffeomorphic to S^6, $S^2 \times S^4$ or $SU(3)/T^2$.

Proof. Part (a) has already been proved in Proposition 4.1. For part (b), from Proposition 4.8, we know that $M^* = S^3$ and there are at most 3 singular orbits, all of which have $U(1)$-isotropy. There has to be at least one singular orbit, since the fixed point set $M^{U(1)}$ cannot be empty. We then discuss the 3 cases of 1, 2 or 3 singular orbits separately.

Case 1: There is only one singular orbit. Then by the slice theorem a tubular neighborhood of the singular orbit is $V = SU(2) \times_{U(1)} D^4 = (SU(2) \times D^4)/U(1)$, where D^4 is a 4-disk and $U(1)$ acts on D^4 via the Hopf action on $\mathbb{C}^2$. V is a linear D^4-bundle over $SU(2)/U(1) = S^2$, with boundary $\partial V = S^3 \times S^3/S^1 = SO(4)/SO(2) = T^1S^3 = S^3 \times S^2$. Linear D^4-bundles over S^2 are classified by $\pi_1(SO(4)) = \mathbb{Z}/2\mathbb{Z}$, and if the bundle is non-trivial then its boundary is the unique nontrivial S^3-bundle over S^2. Thus V is a trivial D^4-bundle over S^2. Moreover, we claim that the slice action by $SU(2)$ on $\partial V = S^3 \times S^2$ is group multiplication on the S^3-factor and trivial on the S^2-factor. To see this, note that the identification $S^3 \times S^2 \cong T^1S^3$ is given by $(p, v_e) \mapsto (p, pv_e)$, where $v_e \in T_eS^3$ and pv_e is quaternion multiplication. $SU(2)$ acts on T^1S^3 via $a.(p, pv_e) = (ap, apv_e) \mapsto (ap, v_e) \in S^3 \times S^2$, and thus it only acts on the S^3-factor.

The complement U of V is an $SU(2)$-bundle over D^3, which has to be the trivial bundle $SU(2)\times D^3 = S^3\times D^3$ with $SU(2)$ acting only on the first factor. Thus M is the gluing of $U = S^3\times D^3$ and $V = S^2\times D^4$ along their common boundary $S^2 \times S^3$ via an equivariant gluing map $f : S^3 \times S^2 \to S^3 \times S^2$. f has to take on the form

$$f(p, q) = (p \cdot g(q), \phi(q)),\ (p, q) \in S^3 \times S^2,\ g : S^2 \to S^3,\ \phi \in \mathrm{Diffeo}(S^2).$$

Since $\pi_2(S^3) = 0$, g is null-homotopic. Note that $\mathrm{Diffeo}(S^2)$ deformation retracts onto $O(3)$ by [Sma59], which has two connected components. Thus there are only 2 homotopy classes of f, depending on whether ϕ is orientation-preserving or reversing. Note that there exists an equivariant orientation-reversing diffeomorphism of $U = SU(2) \times D^3$ given by $(g, p) \mapsto (g, -p)$. If f is orientation reversing, we change the orientation on U equivariantly so that f becomes orientation-preserving. Thus up to change of orientation f is homotopic to the identity map, and $M = U \cup_f V$ is equivariantly diffeomorphic to S^6 .

Case 2: There are two singular orbits. Again a tubular neighborhood V of each singular orbit is $V = SU(2) \times_{U(1)} D^4 = S^2 \times D^4$, and M is the gluing of the 2 copies of V along their common boundary $S^2 \times S^3$ via f. Up to a change of orientation of V, f is homotopic to the identity. Thus the resulting manifold is $S^2 \times S^4$.

Case 3: There are three singular orbits. A neighborhood V' of the singular part is the union of three copies of $S^2 \times D^4$ as in the previous cases. The principal part U' of the manifold is a principal $SU(2)$-bundle over S^3 minus 3 points, which is classified by the homotopy classes $[S^3 \setminus 3\text{ pt},\ \mathrm{B}SU(2)]$, where $\mathrm{B}SU(2)$ is the classifying space of $SU(2)$. Note that $S^3 \setminus 3$ pt deformation retracts onto $S^2 \vee S^2$, and that $\mathrm{B}SU(2)$ has trivial π_1 and π_2. Thus $[S^3 \setminus 3\text{ pt},\ \mathrm{B}SU(2)]$ is a singleton and U' is diffeomorphic to $SU(2) \times (S^3 \setminus 3\text{ pt})$. Thus $\partial U' = U' \cap V'$ is diffeomorphic to three copies of $S^3 \times S^2$. M is the gluing of U' and V' along $U' \cap V'$ via three copies of f. Each copy of f could be orientation preserving or reversing. We fix the orientation on U', and change the orientation of a component of V' if the corresponding gluing map is orientation-reversing. In conclusion, up to change of orientation there is only one homotopy class of the gluing map and thus only one diffeomorphism class of M. From Example 3(a) described in Section 5, we know the flag manifold $SU(3)/T^2$ admits such an action, thus $M = SU(3)/T^2$.

□

Remark 4.11. The $SU(2)$-actions on S^6, $S^2 \times S^4$ and $SU(3)/T^2$ in Cases 1, 2, 3 are all realizable. On S^6 it is the triple suspension of the Hopf action on S^3. On $S^2 \times S^4$ it is the diagonal action where $SU(2)$ acts as $SO(3)$ on S^2 and acts on S^4 as the suspension of S^3. On $SU(3)/T^2$ it acts via left multiplication. We do not know though whether $S^2 \times S^4$ admits a metric with positive sectional curvature invariant under the $SU(2)$-action. The $SU(2)$ actions on the 6-sphere in Case 1 and on $SU(3)/T^2$ in Case 3 preserve positive curvature.

We now summarize the claim about the Euler characteristic.

Theorem 4.12. *Let $M = M^6$ be a 6-dimensional closed simply connected Riemannian manifold of positive sectional curvature such that $SU(2)$ or $SO(3)$ acts isometrically and effectively on M. Then the Euler characteristic $\chi(M) = 2,\ 4,$ or 6.*

Proof. First recall $M^* = B^3$, S^3 or B^4 by Theorem 3.1, and the case of B^4 has be resolved in Theorem 3.1. When $M^* = B^3$, Theorem 1.1(a) reduces to Theorem 4.2((e)). When $M^* = S^3$, from Proposition 4.6 and Proposition 4.8, singular orbits are all isolated whose number is at most 3. In M^{S^1} each singular orbit contributes to 1 or 2 S^1-fixed points. Thus M^{S^1} is a finite set of at most 6 points. Hence $\chi(M) = \chi(M^{S^1}) \leq 6$. By Theorem 2.4, $\chi(M) = 2,\ 4,$ or 6. □

Collecting all the results, we see that the unsettled cases are described in Figures 2-4.

5. Explicit Examples of G-actions

In this section we list all known examples of isometric $SU(2),\ SO(3)$-actions on the known examples of positively curved 6-manifolds, namely S^6, $\mathbb{CP}^3$, $SU(3)/T^2$, $SU(3)//T^2$, and depict the stratification of M^*. For S^6 and $\mathbb{CP}^3$ we list all linear actions. For the known positively curved metrics on $SU(3)/T^2$ and $SU(3)//T^2$, the full isometry group was determined in [GSZ06] and one easily sees that the only isometric actions are the ones described below.

The stratification of M^* for each action is depicted in the corresponding picture.

1. Actions on S^6. Note that all known actions on S^6 are classified.
The following examples have fixed point cohomogeneity at most 1:

(a) **Figure 5a**. $G = SU(2)$ acts on the first 4 coordinates of $S^6 \subset \mathbb{R}^7$ via the realification of $\mathbb{C}^2$ and fixes the last 3 coordinates. ∂M^* consists of fixed points, and the interior has trivial isotropy. Actions of this type are fixed point homogeneous.

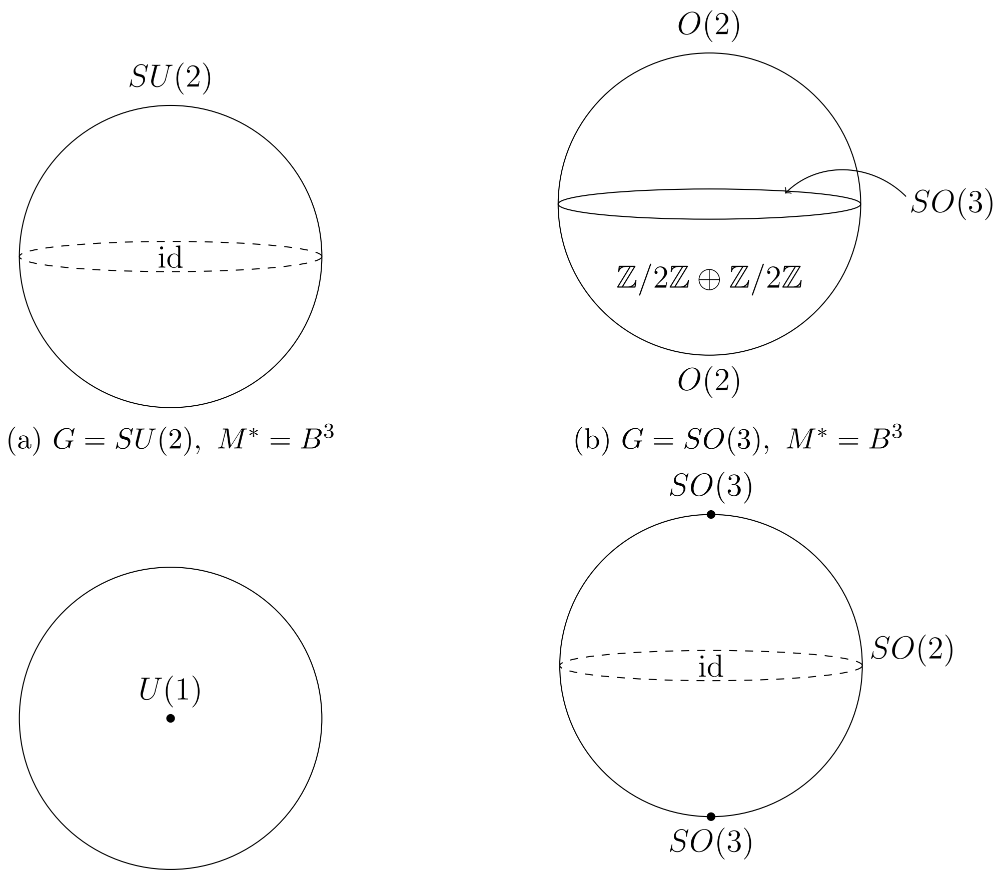
(b) $G = SO(3)$, $M^* = B^4$. G acts on the first 3 coordinates of $S^6 \subset \mathbb{R}^7$ via rotation and fixes the last 4 coordinates. ∂M^* consists of fixed points, and the interior consists of principal orbits with $SO(2)$-isotropy. This is the only case with $\dim(M^*) = 4$. Actions of this type are fixed point homogeneous.

(c) **Figure 5b**. $G = SO(3)$ acts on the first 5 coordinates via the unique 5-dimensional real representation of $SO(3)$ and fixes the last 2 coordinates. The equator of ∂M^* consists of fixed points, and the two boundary faces corresponding to the two open hemi-spheres have $O(2)$-isotropy. The interior of M^* consists of principal orbits with isotropy $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Actions of this type have fixed point cohomogeneity one.

The remaining actions on S^6 are as follows:

(d) **Figure 5c**. This action is given by $A(\vec{x}, \vec{y}) = (A\vec{x}, A\vec{y})$, $A \in SU(2)$, $\vec{x} \in \mathbb{R}^4$, $\vec{y} \in \mathbb{R}^3$, $(\vec{x}, \vec{y}) \in S^6$. The action on the $\vec{x}$-component comes from the real 4-dim irrep of $SU(2)$, i.e. the realification of the standard $SU(2)$-action on $\mathbb{C}^2$, and the action on $\vec{y}$-component comes from the standard $SO(3)$-action on $\mathbb{R}^3$. Actions of this type are classified. See Theorem 1.2.

(e) **Figure 5d**. This action is given by $A(\vec{x}, \vec{y}, z) = (A\vec{x}, A\vec{y}, z)$, $A \in SO(3)$, $\vec{x}, \vec{y} \in \mathbb{R}^3$, $z \in \mathbb{R}$, $(\vec{x}, \vec{y}, z) \in S^6$. Actions of this type are classified. See Theorem 1.3.



(a) $G = SU(2)$, $M^* = B^3$

(b) $G = SO(3)$, $M^* = B^3$

(c) $G = SU(2)$, $M^* = S^3$

(d) $G = SO(3)$, $M^* = B^3$

FIGURE 5. $M = S^6$

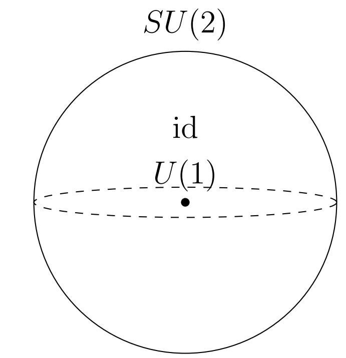
2. Actions on $\mathbb{CP}^3$:

(a) **Figure 6a**. A linear $SU(2)$-action on $\mathbb{CP}^3$, acting on the first 2 homogeneous coordinates and fixing the last 2 homogeneous coordinates. $M^* = B^3$. $\partial M^* = S^2$ consists of fixed points, and the interior minus the center has trivial isotropy. The center has $U(1)$-isotropy, represented by $[x, y, 0, 0] \in \mathbb{CP}^3$. Actions of this type are fixed point homogeneous.

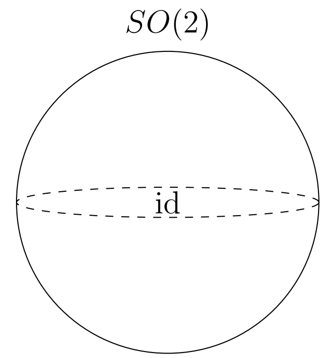
(b) **Figure 6b**. This action is induced from one $SU(2)$-action. Let $A \in SU(2)$ act on $\mathbb{CP}^3$ via $A(\vec{x}, \vec{y}) = (A\vec{x}, A\vec{y})$, $\vec{x}, \vec{y} \in \mathbb{C}^2$. This action is ineffective since $-Id \in SU(2)$ acts trivially, thus descends to an $SO(3)$-action. The interior of M^* consists of principal orbits, and ∂M^* consists of singular $SO(2)$-orbits.

(c) **Figure 6c**. This action is given by $A(z_1 : z_2 : z_3 : z_4) = (A(z_1 : z_2 : z_3)^T : z_4)$, $A \in SO(3)$, $(z_1 : z_2 : z_3 : z_4) \in \mathbb{CP}^3$. $M^* = B^3$. The interior points minus a line segment correspond to principal orbits, the boundary 2-sphere minus 2 points correspond to $SO(2)$-singular orbits, the 2 poles on the boundary correspond to a fixed point and an $O(2)$-orbit respectively, the center corresponds to an $SO(2)$-orbit, and a line segment in the interior corresponds to $\mathbb{Z}/2$-orbits connecting the $O(2)$-orbit and the center $SO(2)$-orbit.

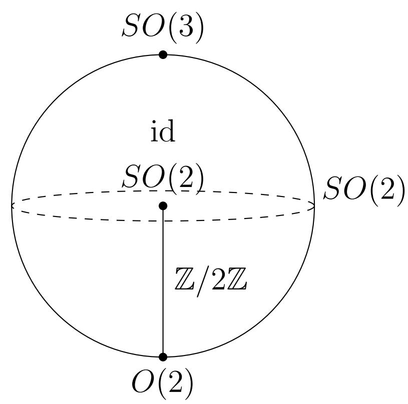
(d) **Figure 6d**. The irreducible representation of $SU(2)$ on $\mathbb{C}^4$ induces an action on $\mathbb{CP}^3$, which is ineffective with kernel $\mathbb{Z}/2\mathbb{Z}$ and descends to $SO(3)$. For this $SO(3)$-action on $M = \mathbb{CP}^3$, $M^* = S^3$, the principal isotropy is trivial, and there are two singular orbits with isotropy $SO(2)$. The exceptional orbits are drawn in the picture.



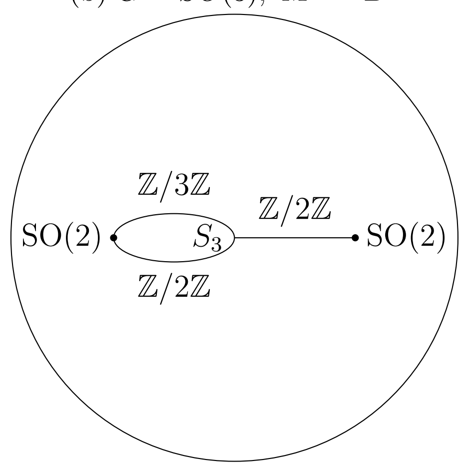
(a) $G = SU(2),\ M^* = B^3$



(b) $G = SO(3),\ M^* = B^3$



(c) $G = SO(3),\ M^* = B^3$



(d) $G = SO(3),\ M^* = S^3$

FIGURE 6. $M = \mathbb{CP}^3$

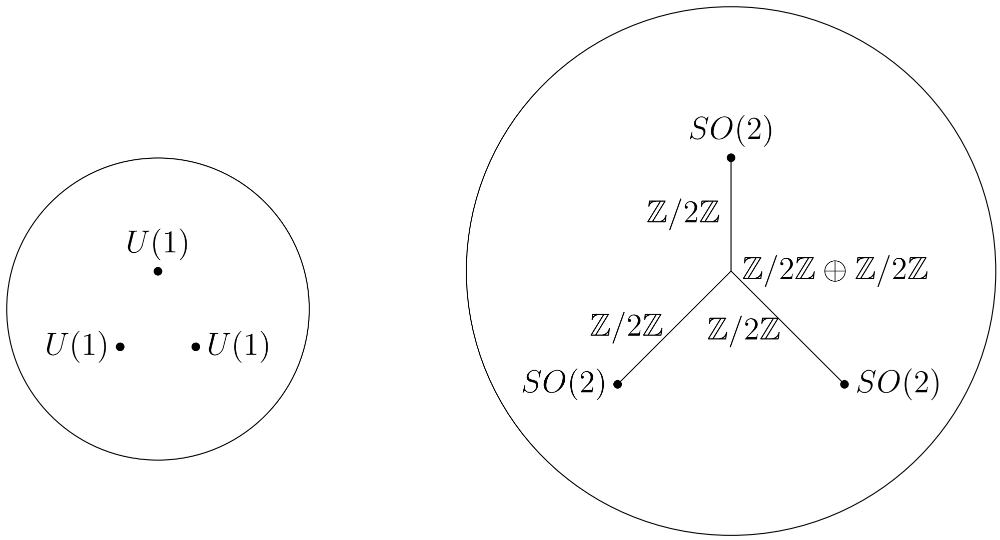
3. Actions on $SU(3)/T^2$:

(a) **Figure 7a**. This action is given by the left multiplication of $SU(2)$ on $SU(3)/T^2$, i.e., $A \cdot (gT^2) = AgT^2,\ A \in\ SU(2),\ gT^2 \in SU(3)/T^2$. Here we view $SU(2)$ as a subgroup of $SU(3)$ and the multiplication Ag takes place in $SU(3)$. The orbit space is a 3-sphere with 3 singular orbits with $U(1)$-isotropy corresponding to the matrices

$$Id,\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The principal isotropy is trivial. Actions of this type are classified. See Theorem 1.2.

(b) **Figure 7b**. This action is given by the left multiplication of $SO(3)$ on $SU(3)/T^2$, similar to the above example. $M^* = S^3$, and there are three singular orbits. The orbit strata are indicated in the picture.



(a) $G = SU(2),\ M^* = S^3$ (b) $G = SO(3),\ M^* = S^3$

FIGURE 7. $M = SU(3)/T^2$

4. An action on $SU(3)//T^2$. Figure 8. Recall that the description of the biquotient is given by $SU(3)//T^2 = (z, w, zw) \setminus SU(3)/(1, 1, z^2w^2)^{-1},\ z, w \in S^1$. $G = SU(2),\ M^* = S^3$. $SU(2)$ acts from the right as the first 2 block of $SU(3)$, commuting with the T^2-action. A computation, using Mayer-Vietoris sequence, shows that G-spaces of this type have the same cohomology groups as $SU(3)//T^2$, that is, $H^0 = H^6 = \mathbb{Z},\ H^2 = H^4 = \mathbb{Z} \oplus \mathbb{Z},\ H^{2i+1} = 0$.

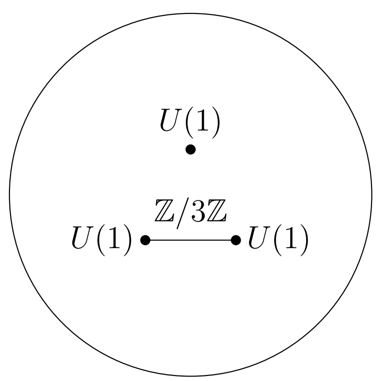


FIGURE 8. $G = SU(2),\ M^* = S^3$

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